This book is with tight Binding

A study of the plight of nations that do not pay their debts

By

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God is not sinless; he created borrowers

Bulgarian Proverb

IN·LUCE·
VERITATIS

Boston

American Unitarian Association

1912

TO

EDWIN GINN

PREFATORY NOTE

Some years since, I began a study of the Eugenics of War, the hereditary effects of the systematic extermination by war of the bold and strong among the yeomanry of the nations of Europe. The first results of this study I set forth in two little books, "The Blood of the Nation" and "The Human Harvest." I soon found it necessary to consider also the "Euthenics" of War, the non-hereditary effects of the financial impoverishment of the rank and file of the people by the cost of war and war armament. This little book is a preliminary survey of the elements involved in this subject. There is a Persian proverb, "He who knows will never tell." In this lies my justification for venturing into a field in which first-hand knowledge is largely out of my reach. I am indebted to Mr. E. Alexander Powell for the phrase, "The Unseen Empire." I am also under obligations to my colleagues, Professors Edward Benjamin Krehbiel, Alvin Saunders Johnson, Payson Jackson Treat and Albert Léon Guérard for suggestions of various kinds. I am also indebted to Mr. Arthur W. Allen of the World Peace Foundation for the tables in the appendix, showing the record of debt and expenditure. Lastly, I am under deep obligations

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to my wife, Jessie Knight Jordan, for constructive and critical work on the manuscript.

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UNSEEN EMPIRE

I. INTRODUCTION

In this book I have tried to tell in part the story of the bondage of the nations due to the cost of war and of war preparation. I have tried to show that civilized nations are one and all in their degree under the dominion of a power stronger than Kings or Parliaments, more lasting than Armies or Navies, that is, the Unseen Empire of Finance. I have tried to show that this mastery is not now in the hands of individual men, however powerful, but that it has passed over into an impersonal Empire of Debt. I have tried further to illustrate "Johnson's law of waste,"[1] to the effect that military expenditures among competitive nations expand in peace or in war as wealth expands, "by the law that war shall consume the fruits of progress," and, finally, to show a way in which our nation, at least, may possibly escape the operation of this law.

I have ventured to believe that Johnson's law is dependent on a lack of continuous purpose in popular governments, and that conditions may be changed by the growth of a robust public opinion opposed to war and debt, and by the extension of treaties of arbitration, which, while dependent on public opinion, yet serve to clinch and hold it in right channels. However great the burden of

[1] Professor Alvin S. Johnson, *The Expansion of Military Expenditures.* International Conciliation, XLI, p. 9.

debt, it is in our hands to shift it. As Sir Edward Grey has said in this connection: "The door of our prison is locked on the inside." The way out lies in the unprejudiced survey of the whole situation on the part of a civilian commission of high minded statesmen who will ascertain the real needs of the people in the line of national defense, regardless of pressure arising from personal interests, from professional ideals of military perfection and from the tendency to follow blindly the fashion set by the "Powers" of Europe. There is no final peace until the civilized nations cease to stand as "Powers" rated according to their capacity to exercise external violence, becoming "States" in the moral union of the world, each one, large or small, being primarily a district of legal and political jurisdiction, not a center of physical force. And at the end I have hoped to make it clear that war debt, the overlordship of the Unseen Empire, the "war scare," and secret diplomacy are all of them necessary stages in the passing of war.

In this book there is no discussion of the existence or the effects of the "money-trusts," international or otherwise, nor is there any account of the various associations, national or international, for purposes of industrial exploitation. The relation of financial matters to war armament and national debt alone concerns us.

I wish to say at the outset that my thesis involves no criticism of the men who compose our

army or who man our ships. I believe, however, that the time has come to cast behind us the thought of war as a possibility in national affairs, and to devote our money and our energies toward more real and immediate needs. We have better weapons than the sword, a more powerful national defense than warships.

I hold in the highest esteem the character and influence of our national academies at West Point and Annapolis. These stand among the best schools of engineering in the world, and none excels them in that discipline of self-restraint which is the foundation of character. Among the officers of our army and navy are many who do most effective work for peace. They recognize the wickedness and futility of avoidable war. Those who know war best realize that there can be no worse calamity.

A typical utterance of a brave man, a statesman as well as a soldier, is this of General Carl Schurz after Gettysburg: "There are those who speak lightly of war as a mere heroic sport. They would hardly find it in their hearts to do so had they ever witnessed scenes like these and thought of the untold miseries connected with them that were spread all over the land. He must be an inhuman brute or a slave of wild unscrupulous ambition who, having seen the horrors of war, will not admit that war brought on without the most absolute necessity is the greatest and most unpardonable of crimes."

II. THE UNSEEN EMPIRE OF FINANCE

In this chapter I try to set forth briefly the story of the rise of the "pawnbrokers" of the world, into whose hands the commerce and industry of the nations have been given as pledges for purposes of war.

It is true that many of these men and the houses they have founded have become prominent as bankers and as leaders in commercial and industrial enterprise. Many of them also have been eminent for personal virtue and as benefactors to their kind. Nevertheless, one and all have laid the foundations of their fortunes in "pawnbroking," that is, in ministering to the demands of spendthrift nations.

A Pregnant Epoch

One of the most momentous periods in world-history was that of the early nineteenth century. This epoch was marked by the coming together of growing forces in civilization, all related in one way or another to the rise of democracy, itself a cause as well as an effect of the passing of civil war.

First of these stands government by the people, as opposed to government by the King and the King's favorites. Next comes the movement of scientific research which in turn gave the impulse toward mechanical invention.

Invention changed the nature of war, as it changed industry and commerce. It made war operations vastly more effective as well as vastly more expensive. Finally, we have the rise of an international system of finance, strong enough and with ramifications wide enough to take whole nations into pawn, and always at hand statesmen ready to pledge the future to any extent in the interest of national glory.

Constitutional Government and Deferred Payment

Constitutional Government gives stability which makes possible deferred payments on a vast scale. The Kings of Old had to pay on the spot. Their credit was bad. They were forced to make their way [illegible]any devices. Among these was extortion[illegible]rm of which was the "patriotic loan," enf[illegible]th the prison as the alternative. Otherwi[illegible]y depended on fawning, bluster, sale of f[illegible]rs, debasement of coinage, issue of paper money, "squeezing" of taxes, and other methods characteristic of the absolute monarchy. *"L'état, c'est moi,"* "I am the State," was the declaration of Louis XIV, and the State, which was the King, borrowed money at a great disadvantage. But a Parliament could bind the whole nation. It could borrow money it never expected to pay. It had only to keep up the interest charges. Thus the debt of Republican France today exceeds many times the largest bor-

rowings of Louis the Magnificent. Even the interest charges alone approach the high water mark of the royal loans of the Eighteenth Century.

National Debt

In the theory of William Pitt, premier of England, the source of authority lay in the people. The men of England owned England and were responsible for its present welfare. "The only source of authority under Heaven," wrote Cromwell, "is the consent of the governed." But in Pitt's view, the owners of England were the people actually alive at any given time. The past had no stake in it; the future had acquired no interest. Therefore, if the men of Great Britain chose to mortgage their nation to secure some great present good, it was their right. Thus immense sums were borrowed and expended in compassing the downfall of Napoleon, and the national debt of Great Britain mounted up to the undreamed-of sum of nearly £800,000,000, a sum which has never been repaid, will never be repaid, can never be repaid so long as the natural growth in national wealth, due to peace, invention and commerce, is all swallowed up by the incredible burden of armament. With the device of the National Debt, as Goldwin Smith observes, "Pitt removed the last check on war." War is no longer limited by the exhaustion of the combatants, but may be continued at the expense

of future generations so long as international "pawnbrokers" are willing to cash the bills drawn against the future.

It is said that Pitt's last words, with the national debt in mind, were these, "Oh, how I leave my country."

Science and War

In the nineteenth century, mechanical invention, constantly active, supplanted the wooden frigate of a hundred years ago by Dreadnaughts and Superdreadnaughts, gigantic floating forts, each costing many times an emperor's ransom, and each new device tending to send all vessels of earlier make to the junkheap. Twelve millions of dollars may now be spent on a single ship and every feature of its upkeep is costly in the same proportion. Like progress has been made in the art of ship-destroying. Shore guns, mines, torpedoes now forbid the approach of a warship to a hostile port. Furthermore all ships and fortresses are already threatened from the air. All appliances of war have steadily increased in effectiveness and in cost, in sad parallelism with the applications of science in other directions. As the cost of war expanded, the need for more money in all warlike nations grew likewise. No nation was willing to be thought not warlike. Thus, more and more, the statesmen of the day were eager to pledge the future of the nation for immediate results.

And as all persistent demand is met by supply, we have as a necessary result the rise of the great "pawnbroking houses" of the world, the first and most powerful of these being the great house of Rothschild.

The Farmer and His Burden

In a French Journal more than a hundred years ago there was published a cartoon. A farmer was plowing in the field and on his back he bore a frilled marquis of the old régime, tapping his dainty snuff-box. A century later, in Paris, was published another cartoon, representing again the burden of France. The farmer still plowed in the field, but now, on his back, was a soldier armed to the teeth and on the soldier's back was borne a money-lender.

"The Peace of Dives"

In his poem, "The Peace of Dives," Rudyard Kipling has cleverly told the story of the way in which Dives, the Rich Man, was allowed to bring peace among the nations, as a condition of his release from his place in Torment. In brief, Dives came forth and went abroad through the earth, selling to the Kings and the Nations the costliest of toys. He sold them Sea-power and Land-power and Imperial Dominion and all Pomp and Circumstance. For these things, so attractively offered, the Kings and the Nations "pledged their flocks and farms" and he bound them hand

and foot in the maze of debt, until even the mightiest of them were no longer fit to fight. "They had pawned their utmost trade, for the dry, decreeing blade," but the blade once in their hands, they had no longer strength to use it. Then Satan appears, but with all his deadliest magic, greed and fear and hate, he found it impossible to stir up the nations to war. For Dives had "trapped them, armoured into Peace." So bound hand and foot and peaceful by force of debt, they were ready to be laid at the feet of the Lord.

The House of Rothschild

It is a true story, that tale of Kipling, but it is a parable and thus needs a bit of interpretation. The name of this man, we may understand, was not Dives. In the beginning it was Mayer Amschel, but as time went on, his descendants continued his work, and with them were many associates. He was not a wicked man, as the original Dives was reputed to be, not even a rich man at first, but sturdy, honest and intelligent. He was not in "Torment." He lived in a narrow, sharp-gabled seven-story house in the Ghetto of Frankfort-on-the-Main. On the front of this house in the old days swung a pawnbroker's sign of the green shield. Later it was repainted and in Mayer's time it was a red shield, "zum rothen Schild." And as time went on, all those who lived in the house received the family name of

"Red Shield," in German, "Rothschild." But to Mayer's family alone was granted the control of the Earth.

The story of the rise of the house of Rothschild is romantically interesting, but its details need not concern us here. Mayer Rothschild was the friend of William IX, Landgrave of Hesse Cassel and through his adjustments the "Hessian" troops entered the British service in the Revolutionary War. This was the first step towards fortune. Others came later, for the Landgrave's gold and his own were subsequently loaned out at a profit to suppliant nations. So from being a local pawnbroker, Mayer became the "Uncle" of Kings, and his worth and power were recognized among the financiers of Europe. Nathan Rothschild, his third son, greatest of world financiers, extended his father's methods to other lands. Establishing himself in London, he placed his brothers in Frankfort, Paris, Naples and Vienna. It was his assertion that while one bank in one capital might fail or might be borne down by a national calamity, five banks in five capitals, all working together, could amply guarantee one another.

Nathan Rothschild was with Wellington at Waterloo. A system of swift messengers, horses and carrier pigeons bore secret news from him (Nathan) in the field to the bank in London, in advance of all other returns.

At once its clerks bought up from scared bond-

holders, at a great discount, a large part of the British National debt, which operation established then and there the supremacy of the House of Rothschild.[1] This supremacy it still holds in so far as it may choose to exercise it.

According to authentic records,[2] the Rothschilds made a large broker's profit each of the four times they handled the £80,000,000 of gold which they bought for use in Wellington's campaign:

1. On the sale of the gold to Wellington.
2. On the sale of Wellington's paper.

[1] "That the House of Rothschild with its branches had an open sesame upon the purse-strings of Europe for half a century, is a fact. Nations in need of cash had to apply to the Rothschilds. The Rothschilds didn't loan them the money—they merely looked after the details of the loan, and guaranteed the lender that the interest would not be defaulted. Their agencies everywhere were in touch with investors.

"For their services the Rothschilds asked only a modest fee—a fee so small it was absurd—a sixteenth of one per cent., or something like that. The bonds were issued and offered at par. If they would not sell at par, they were placed on 'Change and sold at what they would bring. What wasn't taken by the public, brought, oh, say around seventy-five. Unkind people say that the Rothschilds beared all bonds which they, themselves, desired to buy. It wasn't their fault if Leopold's credit was bad,—mein Gott im Himmel!

"It is safe to say that there is but one government in the world that has not at some time or another from 1815 to 1870 courted the Rothschilds with intentions." (Elbert Hubbard. *A Visit to the Home of Mayer Anselm Rothschild.*)

[2] Jewish Encyclopedia.

3. On the repurchase of gold from Wellington.

4. On the sale to Portugal of the gold bought back of Wellington.

Later came "bear" operations looking to the purchase of the British bonds, and the crowding out of rivals (Baring: Goldschmid) in subsequent deals of a similar sort. The sum now owned by the House of Rothschild has been estimated at $2,000,000,000 net. The properties controlled are manifold. According to Mr. E. A. Powell, "they hold one hundred millions of dollars of American securities alone. They own large estates in Great Britain, Germany, Austria, and France, cotton factories at Manchester, cutlery establishments at Sheffield, ships on the Clyde, warehouses in London and Liverpool, gardens near Paris, castles on the Rhine and villas on the Riviera, mills along the Maas, gold mines in California, statues in Rome, dahabiyehs on the Nile, plantations in Jamaica, shawls in India, rubies at Teheran, tobacco fields in Virginia, forests in Siberia, towns in Australia. They call themselves merchants as well as bankers, and in the largest sense they are both." And they are pawnbrokers as well, still "in the largest sense."

The House of Rothschild adopted early these rules of management:

1. The different banks should each act in the common interest, regardless of the purposes of the nation in which it might be placed.

2. They should never deal with unsuccessful people.
3. They should not demand excessive profits.
4. They should never "put all their eggs in one basket."
5. They should always be prepared to sell out quickly in case of prospective failure.
6. They should take advantage of all help to be gained from the press.[3]

Banking and "Pawnbroking"

I have spoken of the early loans of the House of Rothschild as "pawnbroking"; I may for a moment digress to insist on a certain distinction between banking and "pawnbroking."

Banking, properly speaking, deals with "going concerns." It is a provision by which free or idle money may be gathered together and converted into active capital. Through the banker, money on deposit is placed in the hands of those who by industrial or commercial enterprise can make it grow.

"Pawnbroking," broadly speaking, deals with failure or waste. Its usual function is to afford means for some act of extravagance, or escape from some complication of past folly or misfortune. The extravagance, folly and misfortune of nations is summed up in war. Pawnbroking

[3] And at one time, a seventh rule existed: They should not lend to Russia, so long as Jews were persecuted in that country.

among nations thus concerns itself mainly with past war or future preparation, in either case withdrawing the revenues concerned from all productive use.

As there are many nations, ruled by statesmen of the day, ready to sacrifice the future for the present, as no protecting deity watches over their financial operations, and as there exists no official check to national debt, it is clear that the business of the international pawnbroker may be a profitable one. At the same time one must know where to stop. To guard over waste and folly is no sinecure. The cream of the business of international pawnbroking has been now skimmed off, later loans often lowering the values of earlier ones, and in general only weak states in desperate luck are eager to pledge their future revenues.

The Unseen Empire of Finance

Up to the middle of the nineteenth century the great House of Rothschild, with its branches in five leading capitals of Europe, held almost a monopoly of national loans and of the control implied by these loans.[4] Later, with the passing of

[4] I am informed by a friend, a leading banker, that the House of Rothschild to-day has largely withdrawn from that type of banking which deals with war and war debts. They are now rarely at the head of syndicates undertaking national loans, leaving this to the large banking corporations in the various money centers and only occasionally participating in the operations of these syndicates. The house is now known more as a group of capitalists participating in industrial enterprises, discounting

the founders of the house and with the growth of other similar concerns, the leadership of finance becomes more and more impersonal. The individual gives place to a system and the mastery of the Rothschilds is obscured in the rise of "The Unseen Empire of Finance." "The Credit of Europe" and "Das Consortium" are phrases of like significance.

Among the colleagues and rivals of the Rothschilds, their associates in the "Unseen Empire," we may enumerate a few of the most prominent.

In France the House of Péreire is noted for its many enterprises. The "Ligne du Nord," "the Ligne du Sud," the "Compagnie Generale Transatlantique," and the "Crédit Mobilier" are among its creations. The House of Fould was the supporter of Napoleon III. The name of Bischoffheim stands alike for the finance and the philanthropy of France.

In Germany, the name of Bleichröder is forever associated with that of Bismarck and the downfall of the third Napoleon.[5]

paper and profitably investing its surplus funds and those of their old clients, many of whom have in late years drifted to the larger banking operations. Last year the loan to Turkey was negotiated by the Deutsche Bank of Berlin, with Rothschild's and other large German and Austrian banking institutions as participants in the syndicate formed for this loan."

5 "At the treaty of Versailles in 1871, at the close of the Franco-Prussian War, Bismarck made a demand for Alsace and Lorraine, and for an indemnity of 5,000,000,000 of francs. The French representatives, Thiers and Favre,

The name of Camondo, "Citizen of Venice," stands out as that of the supporter of Turkey, "Uncle" to the successive sultans. Through the House of Camondo, Turkey has acquired the debt of $500,000,000, which has been her salvation as a European power. The "sick man of Europe" was kept alive until his debts should be secured. The Goldschmids in London were already great loan agents in the days of Pitt, worthy rivals of the Rothschilds. The Houses of Goldschmid and Stern in London, united by marriage, have made Portugal their own, besides holding large investments in the bonds of other lands.

In Russia, Baron Horace Günzburg[6] of St.

insisted passionately and even with tears that the indemnity named was utterly impossible; that France was wholly unable to pay it or any sum approaching it; and that if counting it had begun at the birth of Christ, it would not yet be finished. To this Bismarck replied, 'But I have provided for that very difficulty,—I have brought from Berlin a little man who begins counting long before the birth of Christ'; and upon this he introduced the Jewish banker, Bleichröder, who found no difficulty in proving that France was so rich that the indemnity asked was really too small.

"Bismarck further claimed Strassburg and Metz for Germany on the ground that there had been twenty-three unprovoked invasions of Germany from France, in days gone by, and not one, save in retaliation, from Germany into France. Germany was henceforth determined to keep the key to her two western doors in her own hands." (Andrew D. White: *Seven Great Statesmen.*)

[6] "Baron Günzburg lives in Russia, where the name of Hebrew is synonymous with persecution. But when the Minister of Finance wants to raise a loan or seeks financial advice he does not send for the Baron to come

Petersburg has been noted as a leader in finance, equally as the promoter of culture among the Jewish people.

Still more widely known and equally respected is the House of Hirsch[7] in Austria, as famous for

to him. He deems it wiser to go to the Baron, for this shrewd, intolerant old man is one of the Masters and every one in Russia knows it from the moujik to the Czar." (E. A. Powell.)

[7] It is said that the Baron Maurice de Hirsch spent, from time to time, over $100,000,000 in charities largely in aid of the people of his race. At the death of his son he said, "My son I have lost, but not my heir; humanity is my heir." Referring to the emigration of the Jewish people from Russia he declared that "Russia would suffer from the loss of her Jews, until to those who remained she would grant civil rights, or else she would fall, as she deserves, the logical victim of her own intolerance." Baron Hirsch held through life one constant aim, that of turning the Jews from the cities to the farms.

"When Baron Maurice died it is said that he left a fortune estimated at anywhere from two million to five hundred million dollars. He controlled—and his heirs still control—the railway systems of all southeastern Europe. Every egg that is laid in the Balkans for European consumption, every yard of cloth, every rifle, every jack-knife that is sold south of the Danube pays a toll to the fortune of the shrewd old Baron. With the vision of a prophet this man wove webs of railways through those districts in the Balkan peninsula which had theretofore been as inaccessible as if they did not exist, and brought a market and employment to those men in skirts and turbans such as had never before stimulated their industry or rewarded their toil." (E. A. Powell.)

"The general impression of Baron Hirsch was of a man with tremendous will power, the instinctive genius and the iron strength of the predestined financier on the grand scale, the kind of man that creates a world-wide trust in the United States.

its great enterprises as for its gains in the field of pawnbroking.

The House of Cassel is intimately associated with the credit of Europe, and even better known

"After he had made his colossal fortune, Baron Hirsch became a prominent figure in the social life of at least three great countries. He had vast estates in Austria, a palace in Paris, a sporting estate in England. He soon became a man whom it was dangerous to cross.

"The Jockey Club in Paris refused him admittance within its doors. He bought the house over their head. In Austria, the stiff traditions of court and society made his social way difficult, but again he was always able to vindicate his position in a dramatic manner, for the then Prince of Wales was glad to accept the invitation to his great sporting parties.

"In London he had innumerable friends, was a powerful person in court circles, and was everywhere received with open arms. And finally when he died, he was seen to have that intense feeling for humanity which is characteristic of his race, who are at once the most materialistic and the most idealistic race in the world, by leaving gigantic sums for charitable purposes, and, above all, for the transfer of the oppressed of his race who wanted to leave the ghettos of Europe for better chances and more liberal institutions in other lands."—(T. P. O'Conner, *Chicago Tribune*, Jan. 21, 1911.)

Baron de Forest of London, a son of Baron Hirsch, has become conspicuous in another field, as a champion of democracy and as an opponent of war and debt as well as in the overlordship of finance which by inheritance belongs to him. Mr. O'Conner continues:

"Here, the idealism of the father is breaking out, but in an entirely new direction. And this idealism, with its gospel of sympathy, above all, for the toilers and the poor, came from a man who might well have been in the other camp—who, if he had been an ordinary man, would undoubtedly have been in the other camp."

in connection with a great engineering achievement in Egypt.[8]

The Sassoons were the Rothschilds of the Orient, —their influence dominant in finance from Yokohama to Bombay.

The names of Mendelssohn, patron of Humboldt, of Montefiore, owner of Australian debt, are better known for their good deeds than for their part in international finance, large as their investments have been. With the names of Werthheimer of Austria, scholar and "Judenkaiser," and Ralli of Athens, "Lord of the Levant," we complete the visible circuit of the leading names in the mastery of Europe. In the same class be longs the house of J. P. Morgan & Co. in America

"It must not be imagined," says Mr. Powell "that these several groups of capitalists ar either rivals or competitors. For what would b the use? They have divided the world amon them, America alone excepted. As a matter o fact, they are all not only friendly, but are al

[8] "The land of Egypt was uneasy and unhappy, for th Lord had withheld the rains in Abyssinia and the Nil ran dry and the cotton crops wilted away under the burn ing African sun. From London came a banker, Cassel b name, and built a great dam across the Nile up near Assuan and the waters poured forth over the parched land even as they had when his ancestor smote the rock, and the blue-shirted fellaheen rose up and called him blessed. They made him a Baronet—whether because he built the dam or rescued the English king from bankruptcy I do not know—and in Egypt he is more powerful than the Khedive and the British consul-general rolled into one." (E. A. Powell.)

lied to one another by so many close ties of blood, marriage and business that it requires but a stretch of the imagination to describe them as a single great group, syndicate, dynasty, empire —The Unseen Empire of Finance.

"To recount the accomplishments of this handful of men is to recount the history of Europe for the last three-quarters of a century. Twice have the Rothschilds saved the Bank of England from suspension; thanks to the ability of old Baron Alphonse, France was enabled to pay the indemnity of five milliards of francs which Germany had imposed in the expectation that it would crush her for a generation. It was on the moneybags of the Foulds and not on the bayonets of his soldiers that Louis Napoleon reached his unstable throne. It was Gerson von Bleichröder who extricated the Prussian Government from its financial difficulties in 1865, played a great part in financing the war of 1870-71, and for his services as financial adviser on the question of the war indemnity had the Iron Cross pinned to his breast at Versailles by the old Emperor William himself. Hirsch opened up the Balkan states to commerce and civilization; Cassel proved himself the latter-day Moses of the Egyptians; Goldschmid, by his gigantic railway schemes, gave Germany a commercial empire in Western Asia."

The Gratitude of Nations

The masters of finance have no need of

blandishments. The coaxing has been all on the other side. The nations who go into debt do so with their eyes open. The indebtedness of Austria to the House of Rothschild brought rank and title to the five brothers at once. That similar gratitude has been felt by other subservient nations is shown by a moment's glance through the biographies of international bankers.[9]

Here we note the names of Baron Nathan Rothschild, the founder of "high finance," Baron Alphonse de Rothschild of Paris, Baron James Rothschild of Paris, Baron Karl Rothschild of Naples, Baron Salomon Rothschild of Vienna, Baron Albert Rothschild of Vienna, Sir Anthony Rothschild of London, Baron Lionel Rothschild of London, Lord Nathan Rothschild of London (the present head of the house), Baron Mayer Rothschild and lastly Baron Willy Rothschild of Frankfort under whose perhaps too conscientious hands the original bank was suspended. Again, Sir Ernest Cassel, Sir Moses Montefiore, Count Abraham Camondo, Baron Sir Edward Albert Sassoon, Baron Maurice de Hirsch, Baron de Forest, Baron Herman Stern, Viscount Stern and Sydney Stern now Baron Wandsworth. Among the Goldschmids, we find Sir Isaac, created Baron de Palmeira by the king of grateful Portugal whose control he shares with his relative, the Viscount Stern. In St. Petersburg, Baron Horace Günzburg and Gabriel Günzburg of Wilna, the

9 As given in the Jewish Encyclopædia.

latter not a baron but granted as a special favor the title of "honorary and hereditary citizen" of Russia. All these and others bear in their degree testimony to the fact that nations and monarchs in distress are not ungrateful.

"Money Power"

In this discussion one may freely admit the sturdy virtues of the founders of the House of Rothschild, and agree that many of their successors in finance have been high-minded men. Nor need we belittle their achievements in the industrial development of the world, nor their generous part in European philanthropy. But this fact remains: Each of these great fortunes was established on national waste, ordered by the people under the forms of constitutional government and in the interest of war.

It is also of course true that the so-called "Money Power" of the world has had many sources and many manifestations not connected with national debt, To follow these in their various ramifications is no part of the purpose of this book.

The recent situation in regard to international finance is summed up by Mr. E. Alexander Powell [10] in a statement which, however, dramatic, can hardly be called an exaggeration. Mr. Powell says:

"The European peoples are no longer under the

[10] *Saturday Evening Post,* June 19, 1909.

Governments of their respective nations. They have passed under another scepter. They have become the subjects of another Power—a Power unseen but felt in palace as in cottage, in Russia as in Spain, by every parent and child, by every potentate and every laborer. No nation on the European continent has any longer an independence that is more than nominal. The political autonomy of every one of them has been surrendered to the will of a despotism before which every kingdom and empire and republic fawns in the most abject subserviency.

"Would the people of Great Britain have you believe that they are free? Great Britain owes a war debt of more than three billion eight hundred millions of dollars. By it she is bound for all time and eternity. She can never pay the debt and she knows it. She never expects to pay it. Of this incalculable sum every inhabitant of the United Kingdom owes something over eighty dollars. Every child born under the Union Jack between Land's End and John O'Groat's is confronted with a bill for a like sum. Such, then is the thraldom of Great Britain—and 'Britons never shall be slaves.' From being the most independent sovereignty that ever existed in the world she has become but a province of the Unseen Empire.

"Is thrifty, industrious France the exception? The French nation, republic though it is, is shackled hand and foot with the chains of her

overwhelming indebtedness—and the money-lords hold the keys. Germany likewise has fallen before their stealthy advance. The German Empire, notwithstanding the bloody victories by which it came into being, notwithstanding its array of battleships and avalanche of armies, notwithstanding the mighty weapon which Bismarck forged and placed in its hand—the financiers picked their steps in the days of that grim old man—dares not, any more than any other European nation dares not, take any important step—to colonize in China or the Cameroon, to build a warship, to dig a canal, to contract for a new rifle, to sign a treaty—without first making petition to the occult Powers of Money who rule and reign from the sandy isles of Friesland to the charcoal-burners' huts of the Böhmer Wald."

The bankers of to-day hold Europe in peace, because, indeed, they hold Europe! [11]

[11] "In the security necessary for international investments lies the prime hope of the world's peace. . . . The Jews, the original missionary people in whom the families of the Earth were to be blessed, have made the millennium possible by the creation of the Bourse." (Israel Zangwill: *Italian Fantasies,* 1910.)

III. THE UNSEEN EMPIRE OF DEBT

In this chapter is given a brief account of the rise and growth of the national debt in certain of the leading nations. In these matters the fault, where fault there is, must be placed mainly on the shoulders of the borrowers. The great masters of finance have played, in general, a waiting part.

War Debt and Other Debt

The extent of the war debt of the individual nations of the world is shown in Table A, as given in the appendix to this volume. These vast sums, in the aggregate, must be described as war debt, because without war all other debts might have been long since paid. It is quite true that much of this money has been borrowed for investment in railways and other productive utilities. Such has been especially the case in France, where about half the debt has had its apparent origin in expansion of national improvements. A considerable part of the debts of Prussia, Italy and Japan are of this nature, as also most of the debt of Canada, Australia, and New Zealand. In some cases these investments have proved unwise or losing ventures, thus adding to the aggregate of inert debt.

Yet it is unquestionably true that by far the greater part of all this debt had its origin in war,

and that in the last fifty years the aggregate repayments on war debt, together with the interest payments on such debt, have greatly exceeded the total amount of debt of non-military origin. In other words, if it had not been for war and war preparations, the natural income of the nations would have easily paid off all indebtedness, including that borrowed for industrial and commercial expansion.

We may further add that debt, whatever its origin, is still debt and has the same general effect of restricting the financial freedom of nations. Again, in all lands, non-military debt, in common with other indebtedness, tends steadily to increase. The nation which seeks to reserve its available funds for purposes of war becomes thus still further embarrassed. Peace revenues should be adequate for peace purposes. The limit of safe borrowing lies in the ability of the investment made to pay its way—to cover its interest charges. The world over, deferred payments of nations have found in war their origin and their excuse. The national debt of the world, when fully analyzed, is war debt pure and simple.

In any discussion of national debt, we must remember that other branches of government are not sinless. The national interest charges are all superimposed on the charges paid by a large system of local and municipal indebtedness. And to this again is added the burden of debt carried by the individual citizen. Municipal debt has in

every land been a factor in municipal ownership of utilities. It may be wise and legitimate to borrow money in an expanding business when the capital taken yields its proper return in interest. A rapidly growing city can count on the reduction or dilution of its burden by an increasing population. But to borrow money to postpone evil days without adequate means to meet the interest is dangerous and demoralizing for municipalities as for men or nations.

There is much to be said for government ownership of utilities, railroads for example. But for a government to buy without paying, or to borrow the money from the original owners to pay for the roads, is merely to wager that government management can make more out of them than private owners can. It may be justified only when it succeeds. On the whole, most borrowings, state or municipal, fail in this regard.

The experience of France in buying the "Ouest" system of railways on these terms is a case in point. The semi-political management now existing is at a disadvantage in every way, and the system has been a source of increasing loss from the time the government acquired it.

Debt of Great Britain

The bonded debt of Great Britain [1] properly

[1] The account of the growth of debt in Great Britain, France, Germany and the United States is chiefly condensed from the *Credit of Nations*, by Francis W. Hirst: Washington, Senate Document No. 579, 1910. See also the diagram in the Appendix of this volume.

begins with the Revolution of 1689. Before that time the King of England, as was common with monarchs generally, frequently raised sums by the pledge of jewels, the mortgaging of temporary revenues or the extracting of loans (not always repaid) from the Jews. In the war of King William III against James II and the King of France, Parliament was ready to pledge the national resources, as the available taxes were not adequate for the operations of war. By thus forestalling taxes, the foundations of the great national debt were laid. Because of the doubtful credit of Parliament at the time, the loan of £250,000 in 1690 could not be had at less than 8% interest. Numerous other loans followed, for which various sources of revenues were pledged, as customs duties and finally "taxes on bachelors, widows, marriages, births and burials." Even as thus secured, the bonds were sold at a heavy discount. The name of "Dutch Finance" was applied in reproach "by old-fashioned people to the various devices for throwing the burden of expenditure on posterity, that were introduced along with William of Orange and 'the Glorious Revolution.' "

In the reign of Queen Anne, lotteries were added to the sources of revenue and an era of official and private speculation culminated in the "South Sea Bubble."

In the reign of George I, quiet and frugality reduced the debt, which had amounted to over

£12,000,000 in the reign of William III and to £52,000,000 in the time of Queen Anne. Other wars raised the figure to £129,000,000, where it stood at the beginning of the American Revolution in the reign of George III. "By this time it was clear that the national debt was advancing at a dangerously rapid rate and the whole of it had been spent on war." The war with the American colonies was still more disastrous from the point of view of finance. The total debt at the end of the war was nearly £250,000,000, and the value of the "funds" or bonded evidences of debt fell with each increase of borrowed money. At every British defeat they fell lower, and scarcely rose with British victory. "They fell and fell; the capitulation of Lord Cornwallis reduced them to 54, and they could scarcely have gone lower if they were to retain any value at all." The lowest point indicated by Mr. Hirst is 47.

The figures above named indicate only the funded debt, which had risen in 1814 after Napoleon's retirement to Elba to £743,000,000. After Waterloo, the unfunded debt is estimated at £60,000,000 and the funded debt at £826,000,000. This gigantic increase, the beginning of what we may call world bankruptcy, was due to the policy of Pitt and his successors, avowedly throwing on posterity the cost of the downfall of Napoleon.

Retrenchment has at different times made re-

duction in this debt, the most extensive and successful effort covering the period between the years 1887 and 1899. This brought the value of 3% consols[2] up to 110. The Crimean war had added £33,000,000 to the debt. Other dangers to credit have arisen from the Imperial Defense Act of 1889, and still later from the constant rise in military and naval expenses.

The political policy which preceded the Boer War caused the steady fall in value of the consols. With this war came a marked depreciation of credit until 1901, when consols stood at 91¾. The Boer War raised the national debt from £635,000,000 (in 1899) to £798,000,000, and "the national savings of thirty-six years of peace were swept away by national borrowings during three years of war." There is, however, a small but regular decrease of the British National Debt which goes on automatically, through the operation of a national sinking fund.

At the present time English consols stand near their lowest point, 76. The effect of this depreciation in government securities may be illustrated by the history of the Birkbeck Bank in London, here compiled from London newspapers of the time.

This bank was noted as one of the most conservative in England, its reserve being largely in consols and other supposed high-grade, low-interest securities. It went into insolvency be-

[2] See note page 32.

cause of the failing price of these securities. It suspended payment on June 8, 1911, with a deficiency of about four hundred thousand pounds. It had, shortly before, weathered a severe run which resulted in the withdrawal of three millions of pounds, little of which had been returned. After the crash, Bernard Shaw appealed to the government to come forward and make good the four hundred thousand pounds, claiming that "it would have paid us as a nation to subsidize the Birkbeck to four times this sum annually had such help been necessary. Now 112,000 people, who, if they had their houses shaken down by an earthquake would have been rescued by the public as a matter of course, are thrown into the most distressing anxiety and threatened with a calamity that will spread far beyond the direct sufferers." Mr. Shaw therefore begs the government to reopen the doors of the Birkbeck Bank and "to give it hopes of such an annual grant-in-aid as will save it from retreating, like the other banks, into the service of the comparatively rich only. There is no class in which the struggle for existence is so warring and incessant as in the class that banked at the Birkbeck. Are they to be abandoned to a calamity which will do several millions' worth of mischief when the yield of about half a farthing on the income-tax would avert it?" It was said in Parliament by an official that the failure of the bank was due to "Lloyd-George Finance." The Chancellor of the Exchequer re-

torted by showing that it was due to the depreciation in the value of securities and especially of the government securities known as consols.[3] "At the time of the Boer War the reserve was more than ample to cover every scrap of depreciation. At that time our securities began to drop away. In 1899 certain securities stood at 112. They are now worth 85½. Certain others stood at 101 in 1899, and now at 84. A factor in this depreciation is the Colonial Security Act."

In Great Britain and Ireland, the local debts are estimated in 1901 at £376,000,000, in 1906 at £560,000,000. These are growing at a rapid rate, but they differ from most national debts in yielding a fair interest return. Nevertheless, the fact of a rapidly growing municipal indebtedness in every city greatly complicates the problem of national debt and national expenditure.

Debt of France

The national debt of France dates from the French Revolution. But the Kings of France, on the theory of absolute monarchy, had been borrowers for long before. The earliest king to raise loans on security was Francis I. He secured money from the city of Paris, and in return alienated certain royal privileges, which

3 Rigidly interpreted, the issue of "consols" (that is, "Consolidated Bank Annuities") is not the borrowing of money directly, but through the sale of "perpetual annuities."

became known as "rentes sur l'Hotel de Ville." The royal debt rose in 1561 to 74,000,000 francs, a sum so large that Catherine de Medici thought to reduce it by the seizure of ecclesiastical properties. The clergy evaded this by a new debt, "rentes sur la clergé," accompanied by various exemptions from taxation. Under Henry IV the debt had arisen to about 337,000,000 livres,[4] which Sully reduced by about 100,000,000 livres. Afterwards borrowing became habitual and under Louis XIV, the prince of borrowers, "the finance ministers had a hard task to supply their master's prodigal magnificence." The tontine annuity became a popular method of raising money for the King. This state of affairs was changed by the great economist Colbert who took active measures to reduce the debt, "acting on his belief that rentes were a most useless and expensive possession of a state. He had no belief in the benefits of credit. In his eyes loans were always made by idle capitalists for unproductive purposes and he looked upon the interest charge as an improper burden on the taxes." This sound view had its practical disadvantage, for war was actually on and under these conditions Colbert found it very difficult to raise any money at all. It is the device of the deferred payment which makes modern war preparation, and therefore modern war, possible to any nation.

With the death of Colbert "all sound manage-

[4] Eighty-one livres tournois equals 80 francs.

ment vanished from French Finance." "Aprés nous le déluge" was a current motto. In 1715, at the end of the reign of Louis XIV, the debt stood near 2,000,000,000 francs. Afterwards, following the operations of John Law and his reckless colleagues, it rose to 2,400,000,000 livres. Then followed a period of bankruptcies and forced loans called "reductions and consolidations of debt." With Turgot, under Louis XVI, came a period of retrenchment and wise management, "neither bankruptcy avowed nor masked by forced reductions, nor by increase of duties, nor by borrowing," a policy which, in Hirst's opinion, might have saved the monarchy of France. But Turgot's successors, Necker and Calonne, continued to borrow and confiscations on a large scale took place under the guise of forced loans and loans repaid in paper "assignats."

In 1793, Cambon created a great body of public debt, in which the king's indebtedness took the modern form of a national debt. It had now risen to a capital value of 3,500,000,000 francs, which was cut down by paper money, confiscations and patriotic loans, in 1797, to 800,000,000 francs. Thus was eased the strain on the government at the expense of the people, causing consternation and bankruptcy far and wide.

Under Napoleon, the issues of inconvertible paper money ceased and loans were avoided so

far as possible.[5] Napoleon hated debt, but his "policy of making war pay its way" imposed very heavy annual burdens on France and still heavier ones on the conquered territories. France thus escaped, however, the burden of permanent debt, and her financial condition under Napoleon was "enviable compared with that of the victorious Government of Great Britain." The aggregate increase of French debt in the time of Napoleon was about 140,000,000 francs. But this figure in no degree represents the money cost of Napoleon's wars either to France or to her willing and unwilling allies.

Under Louis Philippe, the debt steadily rose until it reached the sum of 3,540,000,000 francs. The Second Republic was a period of financial disorder, of "forced loans" and converted obligations, leading to a capital debt of 4,620,000,000 francs, or about $920,000,000.

The Second Empire continued the policy of war (and borrowing) in the Crimea, in Mexico and in Italy. It fell in the Franco-Prussian war of 1870-1872, a struggle unprecedented in European history for cost and waste. The final result was a debt of 9,000,000,000 francs. "The enormous stored-up wealth of France and the recuperative powers of the nation were then won-

[5] "You have supplied and paid the army; you have remitted 30,000,000 francs to the state treasury; you have enriched the museum at Paris with 300 objects, the products of 30 centuries." (Napoleon, 1797, at Bassano. *Address to His Soldiers.*)

derfully displayed." The Germany indemnity of 5,000,000,000 francs was promptly paid by two loans through the French house of Rothschild, and made good by the patient industry of the people. Then followed giant loans in the interest of the "great programme" of schemes of public works instituted by de Freycinet. The guarantee of interest on railways, the purchase of the Ouest system, the expansion of the army and navy, the cost of the war and the Commune brought France's debt in 1908 to the unprecedented figure of 30,161,000,000 francs. This is the largest aggregate burden yet borne by any nation, and its interest charges are double those of Great Britain. It is true, however, that the debts of the other Latin nations are higher in proportion to resources than that of France.

The local indebtedness of France has grown in like proportion. It is estimated in 1906 at 4,021,000,000 francs, having risen from about 60,000,000 francs in 1830.[6]

The national taxes of France including the interest on the public debt, amount to very nearly 5,000,000,000 francs yearly. In addition to this and to the local taxes is added the aggravation of the Octroi or local impost on articles entering the limits of town or city.

[6] For a diagram of the growth of the public debt of France, the reader is referred to the Appendix of this volume.

Debt of Germany

Germany, as a consolidated nation, is the youngest of the Great Powers, her territory having been for centuries the battle ground of her neighbors. Since the establishment of the Empire, her bonded indebtedness has risen by leaps and bounds, for her war and peace expenditures have been on a grand scale. In 1877, the debt stood at 72,000,000 marks. It is supposed, however, that a modest sum, commonly stated as 120,000,000 marks, had been held over out of the total amount received from France in 1873 and still rests in the Julius Tower at Spandau. But against even this, treasury notes to the full amount have been issued by the government.[7] The payment of the indemnity was a heavy burden on France, but its effect on Germany was equally burdensome, for it led to an era of speculation and unregulated production, intensifying those evil results which always follow victory in war.

"Norman Angell" (Ralph Lane) says of this experience:[8]

"The decade from 1870-1880 was for France a great recuperative period, and for Germany,

[7] In the United States, the deposits of gold and silver in the treasury against which notes have been issued are not counted as assets of the government, as understood by us. The "Imperial War Treasure" at Spandau is not the property of the Empire, but of the holders of the notes issued against it.

[8] The Great Illusion.

after a boom in 1872, one of great depression. We know that Bismarck's life was clouded by watching what appeared to him an absurd miracle: the regeneration of France after the war taking place more rapidly and more completely than the regeneration in Germany, to such an extent that in introducing his Protectionist Bill in 1879 he declared that Germany was 'slowly bleeding to death,' and that if the present process were continued she would find herself ruined. In the Reichstag, May 2, 1879, Bismarck said:

" 'We see that France manages to support the present difficult business situation of the civilized world better than we do; that her Budget has increased since 1871 by a milliard and a half, and that, thanks not only to loans; we see that she has more resources than Germany, and that, in short, over there they complain less of bad times.'

"And in a speech two years later (Nov. 29, 1881) he returns to the same idea:

" 'It was towards 1877 that I was first struck with the general and growing distress in Germany as compared with France. I saw furnaces banked, the standard of well-being reduced, and the general position of workmen becoming worse, and business as a whole terribly bad.'

"Trade and industry were in a miserable condition. Thousands of workmen were without employment, and in the winter of 1876-7 unemployment took great proportions and the soup-

kitchens and State workshops had to be established.

"Every author who deals with this period seems to tell the same tale. 'If only we could get back to the general position of things before the war,' says Maurice Block in 1879. 'But salaries diminish and prices go up.' [9]

"In examining the effect which must follow the payment of a large sum of money by one country to another, we saw that either goods must be imported by the nation receiving the indemnity to compete with those produced at home; or the gold must be kept at home and prices rise and so hamper exportation; in the case of the country losing the gold, prices must fall and exports rise. That this, in varying degrees, is precisely what did take place after the payment of the indemnity we have ample confirmation. The German economist, Max Wirth (*Geschichte der Handelskrisen*) expresses in 1874 his astonishment at France's financial and industrial recovery: 'The most striking example of the economic force of the country is shown by the exports, which rose immediately after the signature of peace, despite a war which swallowed a hundred thousand lives and more than ten milliards ($2,000,000,000).' A similar conclusion is drawn by Professor Biermer, who indicates that the Protectionist movement in 1879 was in large part due to the

[9] "La Crise Economique," *Revue des Deux Mondes*, March 15, 1879.

result of the payment of the indemnity, a view which is confirmed by Maurice Block, who adds:

"'The five milliards provoked a rapid increase in imports, giving rise to extravagance, and as soon as the effect of the expenditure of the money had passed there was a slackening. Then followed a fall in prices, which has led to an increase in exports, which tendency has continued since.'

"But the temporary stimulus of imports—not the result of an increased capacity for consumption arrived at by better trade, but merely the sheer acquisition of bullion—did grave damage to German industry, as we have seen, and threw thousands of German workmen out of employment, and it was during that deçade that Germany suffered the worst financial crisis experienced by any country in Europe."

In 1908, the national debt stood at 4,253,000,000 marks, the lion's share of it having been paid out for the army and a steadily increasing percentage for the navy. To these sums should bė added the debts of the individual states, that of Prussia alone having grown at a rate comparable with that of the burden of the Empire. The total debt of the Empire and all the states was, in 1908, 14,362,000,000 marks or about $3,600,000,000, which is not far from the debt of Great Britain. The debt of Prussia arose from 1,965,000,000 marks in 1881, to 7,963,000,000 in 1908. Under the policy of Germany, by

which the expenses of the army in time of peace are met mainly by loans, we may expect the steady increase of the national debt which has been rising of late at the rate of 60 to 100 millions of dollars per year. This is balanced in a degree by the extension of national holdings, especially in the financially profitable state railway system. On the other hand the absorption of German capital in industrial affairs creates temporary embarrassments through the necessity for short-time loans which German banks cannot wholly cover and which are largely drawn from London, Paris and New York. German government bonds range at about 80. As investments, I am told, they are now generally avoided by the banks as undesirable forms of security. The day is past when the war debts of great nations offer especial attractions to those whom in the past they have made "masters of Europe."

The local debt in Germany as in other countries rises steadily with the extension of municipal improvements. The total is given by Mr. Hirst as over 7,400,000,000 marks. It is said that the debts of the German towns combine financial security with a high rate of interest. This is due to their excellent municipal management.

Debt of the United States

In the early days, most of the Colonies had embarrassed their finances by large issues of paper money, at first to meet war emergencies

and afterwards ordinary expenses. These notes fell rapidly in nominal value and they were finally for the most part repudiated.

At the outbreak of the Revolution the same mischievous practice continued on the part of the Continental Congress and of the individual states. The notes lost in value until "not worth a Continental" was a general expression of worthlessness. It is said that Boston was in 1779 "on the verge of starvation: money transactions had nearly ceased and business was done by barter."

In 1787 the Federal Constitution granted Congress "power to borrow money on the credit of the United States," while forbidding individual states to coin money or to emit bills of credit. "The Government of the United States," says Mr. Hirst, "inherited from the states of which it is composed the vicious principle of confounding debt with currency. The crude notion of raising money by debasing the currency whether by adulterating the metal or by issuing an excess of paper has now been relegated to the least civilized and intelligent states of the world. But traditions die hard and the system of propping up credit by currency regulations may still be traced in the laws of the United States." [10]

The national debt once started grew rapidly as in other nations, notwithstanding the influence of wise financiers, notably Albert Gallatin.

In 1816, the bonded debt stood at $127,000,-

[10] Hirst, *The Credit of Nations,* p. 104.

000. In following years of peace this amount was rapidly reduced. In 1835 it stood at about \$34,000,000. Each war which followed was accompanied by increase of debt. The Mexican War added \$49,000,000 to the sum total which stood in 1851 at \$68,000,000. Succeeding governmental extravagance checked its natural reduction so that in 1860 it amounted to \$64,800,000. The Civil War, with its gigantic expenses and depreciated currency, raised it to \$2,773,200,000. The country had been flooded "with short time paper which served the purpose of currency, expanded prices and increased the speculation and extravagance always incidental to war. Temporary obligations falling due in the midst of civil conflict were a source of double vexation to the Treasury Department, which was obliged to conduct a series of refunding operations and at the same time to go into the market to borrow ever increasing sums." [11]

The Confederate states met their expenses almost wholly by treasury notes which served as the currency of the people. These steadily fell in value, until they became worthless and the first issues were repudiated to make way for later ones equally worthless. It was regarded as impossible to carry on war by means of taxes alone. "This," says White, "was a mistake. Except for money borrowed abroad every country pays the cost of a war at the time of the war. All of the

[11] Dewey, *Financial History*, p. 317, as quoted by Hirst.

debts of the confederacy were obliterated at the end of the war. . . . There being nobody else to pay for it, the people of the Confederacy must have paid for it during the time of the war and not a moment later."[12]

After the war, there were many fluctuations in the values of paper money, and several financial panics in which the relation of "greenbacks" and of silver to gold bore a certain part. In 1892 the bonded debt had fallen to $585,000,000. The war with Spain raised it to $1,046,000,000 in 1899. In 1911 it had been reduced to $915,-353,000.

Up to 1900, "it had always been the policy of the government to pay its interest-bearing debts as soon as possible in order to avoid unnecessary burden on the taxpayers." But at present this policy has been more or less cast aside in the interest of military expansion and unwarranted River and Harbor improvements. Old-time economists, occasionally found in Congress, have rarely succeeded in checking the extravagance which the politicians and armanent syndicates demand, and which the people seem able and willing to pay for with borrowed money.[13] "I know but

[12] Horace White, *Money and Banking*, p. 148, as quoted by Hirst.

[13] In connection with Table A in the Appendix, Mr. Arthur W. Allen, author of the table, makes the following explanation of the reasons why the gold and silver certificates are not to be regarded as part of the national debt.

"In the account of the national debt of the United

one way," said Albert Gallatin in 1800, "that a nation has of paying her debts, and that is precisely the same that individuals practice; spend less than you receive, and you may then apply the surplus of your receipts to the payment of your debts."

The local debts of the various states, counties and municipalities, mount to a large figure. In 1902, the sum total is quoted at $1,765,000,000. This with all other items in the record is steadily rising. The aggregate indebtedness of the United States is, however, small in proportion to her resources and their constantly expanding possibilities.

Debt of the Lesser Nations

The bonded debts of the smaller countries of Europe need not be discussed in detail. In gen-

States, given in table A, I do not include $1,461,600,000 in the form of gold and silver certificates. Against these certificates there is held their exact equivalent in gold and silver coin or bullion. This being the case, though these may be technically included in the debt of the United States, they are in reality mere warehouse receipts. From the business point of view they are no more a debt, as a matter of fact, than is a bushel of wheat or a bale of cotton or a piece of furniture, for which a warehouse has issued its receipt. These articles in every case are the property of the person holding the certificate, and the certificate is an evidence of ownership—not a promise to pay. They can by no process be lawfully used for the payment of any other obligation of the warehouse."

An actual obligation, or debt, is involved in the issue of treasury notes or "greenbacks," unsecured promises to pay, to the amount of $346,681,506.

eral, they have followed the lines indicated in the discussion of the debt of France. But the other nations of southern Europe have borrowed even more recklessly than France. The debt of Italy, for example, is nearly half that of France while her resources are only about one-fourth as great. The debt of Spain is nearly one-third that of France, while her wealth is but one-tenth. The debt of Japan, $287,500,000 in 1903 (less than half that before the war with China) since the war with Russia, has risen to about $1,325,000,000. The debt of Persia (about $26,000,000) is due mainly to the benevolent attentions of her neighbors, who, in the words of the *London Times*, "are exercising control over her for her own good." Were she bound hand and foot to the international bankers as Turkey is, she would be regarded as fitted for a career as an independent nation.

The small nations which have given up the struggle and frankly devoted themselves to their own business are more or less definitely free agents. It is true, however, that in most cases only the jealousies of their larger neighbors, the "balance of power," has saved them from absorption. Among these nations relatively free and prosperous,—prosperous because free,—are Switzerland, Denmark, Holland, Belgium, Norway and Sweden. In each of these, the average wealth of the common citizens runs higher in proportion to population than in England and

Germany. This is because their money is largely left where it belongs, in the hands of the people. But in all of these one may trace the influence of the evil example set by their larger neighbors.

Canada owes over $336,000,000, none of it war debt, money invested in "going concerns" of her great railways, the profits of which are largely returned to her people.

New Zealand has little actual war debt. Her various civic experiences she has purchased at a cost of a bonded debt of nearly $346,000,000, a large sum for less than a million people, but their islands are unusually rich in natural resources, and very little of this money, about six per cent only, has gone into the waste of war. This includes the cost of Maori wars and of a small armament.

Australia is a greater nation, less favored by nature, and from the liability to drouth, subject to greater financial risks. She has no war burden save what she has patriotically and needlessly assumed, but she has likewise ventured on costly experiments, the burden of which she has thrown on generations to come.

Relative Cost of War

Meanwhile, during the nineteenth century, the wealth and resources of each of these nations have grown with unexampled rapidity. The increase has been primarily due to mechanical in-

vention, which applied to manufacture and commerce has enormously augmented the effectiveness of the individual man. It has also enabled nations to dispose of their products to better advantage than when all articles produced had to be consumed within a limited district. Through the development of commerce all nations have become neighbors and international trade is now one of the greatest factors in human society. At the same time have come better adjustments in all relations, a better division of labor among men, and in general a condition in which men of all parts of the world may contribute to each other's welfare. The increase in wealth vastly exceeds the increase in national debt, but this fact has been more or less completely neutralized by the gigantic expenditures on war preparation to which nations have been incited by the vastly increased cost of war itself. Enormous as economic growth has been, it is still subject to the check involved in Johnson's law, that with expanding and competing nations, "war shall consume the fruits of progress." The burden of tax and debt bears just as heavily on the toiler to-day as it did a century ago, and this notwithstanding the rapidly growing political importance of the individual. The superior prosperity of certain nations is due to the failure—for various reasons—of expenditure to keep pace with growing resources; in other words, such nations have not been living beyond their means.

"The England of William and Mary," says Professor Johnson, "a great power, conscious of her destiny to command the seas, spent on her army and navy about the same amount of money that Switzerland, protected by treaties and the conflicting interests of the powers, spends on her army to-day. The England of 1775, facing the revolt of her American colonies and confronted by bitter enemies on the Continent, spent on her army and navy a quarter more than Belgium spends to-day on an army that might almost be said to exist for ceremonial purposes alone, since Belgium would be absolutely helpless in case of aggression by a great power.

"Absolutely considered, modern peace is unquestionably vastly more expensive than the wars of an earlier period. True, the wars involved a large destruction of property and loss of life that never figured in the expenditures. But our statistics for times of peace also fail to include the waste of time entailed by universal military service and the waste of ability in organizing so vast an enterprise as an army of peace.

"It is true, of course, that the world to-day is incomparably richer than the world of Pitt or of Marlborough. The per capita burden of modern armaments would have been almost intolerable to the British of the eighteenth century; it would have crushed the taxpayers of France or Germany. It is possible—though not very probable—that the burden of military expendi-

tures signified greater hardship to the average citizen of an eighteenth century state than it signifies to the citizen to-day."

Trusts versus War Debt

The bonded debt of the United States amounts to $915,353,000, a huge sum it is true, but of relatively modest proportions as compared with the debts of Europe. Whether our burden of trusts and privileged interests in America is less weighty than the load of war debt in Europe added to their own burden of privilege (none the less oppressive because of long standing) is a question aside from our discussion. But the war debt of Europe certainly presses there more closely and more persistently, especially on the laboring class, while in America efforts toward the removal of privilege have been more continuous and effective than most similar struggles in Europe. However, as the present generations in both Europe and America have been born to this debt, the burden pressing everywhere and mostly by indirection, the individual man does not feel it acutely. It is like the pressure of the atmosphere, always felt and therefore not recognized. But a release, were it possible, would work enormous social and economic changes in the long run wholly for the better.[14]

Throughout modern history, two of the most effective weapons of tyranny have been the De-

[14] See p. 127, "Economic Difficulties in Disarmament."

ferred Payment and the Indirect Tax. By the latter, the people never know what they pay, by the former all embarrassments are thrown on posterity. The men of to-day are the posterity of yesterday, and they bear on their shoulders all that the nineteenth century has shirked. For debt there have always been a thousand excuses, but there is only one relief. Nations as well as individuals must pay as they go.

The Spendthrift Age

The expanded credit of the world, according to the editor of *Life*, may be likened to "a vast bubble on the surface of which, like inspired insects, we swim and dream our financial dreams. . . . We have long since passed the simple or kindergarten stage of living beyond our incomes. We are now engaged in living beyond the incomes of generations to come."

Let me illustrate by a supposititious example. A nation has an expenditure of $100,000,000 a year. It raises the sum by taxation of some sort and thus lives within its means. But $100,000,000 is the interest on a much larger sum, let us say $2,500,000,000. If instead of paying out a hundred million year by year for expenses, we capitalize it, we may have immediately at hand a sum twenty-five times as great. The interest on this sum is the same as the annual expense account. Let us then borrow $2,500,000,000 on which the interest charges are $100,000,000 a year. But while pay-

ing these charges the nation has the principal to live on for a generation. Half of it will meet current expenses for a dozen years, and the other half is at once available for public purposes, for dockyards, for wharves, for fortresses, for public buildings and above all for the ever growing demands of military conscription and of naval power. Meanwhile the nation is not standing still. In these twelve years the progress of invention and of commerce may have doubled the national income. There is then still another $100,000,000 yearly to be added to the sum available for running expenses. This again can be capitalized, another $2,500,000,000 can be borrowed, not all at once perhaps, but with due regard to the exigencies of banking and the temper of the people. With repeated borrowings the rate of taxation rises. Living on the principal sets a new fashion in expenditure. The same fashion extends throughout the body politic. Individuals, corporations, municipalities all live on their principal.

The purchase of railways and other public utilities by the government tends further to complicate the problems of national debt. It is clear that this system of buying without paying cannot go on indefinitely. The growth of wealth and population cannot keep step with borrowing even though all funds were expended for the actual needs of society. Of late years war preparation has come to take the lion's share of all funds how-

ever gathered, "consuming the fruits of progress." What the end shall be, and by what forces it will be brought about, no one can now say. This is still a very rich world even though insolvent and under control of its creditors. There is a growing unrest among taxpayers. There would be a still greater unrest if posterity could be heard from, for it can only save itself by new inventions and new exploitations or by a frugality of administration of which no nation gives an example to-day.

The Burden of Armament

Nevertheless this burden of past debt, with all its many ramifications and its interest charges, is not the heaviest the nations have placed on themselves. The annual cost of army and navy in the world to-day is about double the sum of interest paid on the bonded debt. This annual sum represents preparation for future war, because in the intricacies of modern warfare "hostilities must be begun" long before the materialization of any enemy. In estimating the annual cost of war, to the original interest charges of upwards of $1,500,000,000 we must add yearly about $2,500,000,000 of actual expenditure for fighters, guns and ships. We must further consider the generous allowance some nations make for pensions. A large and unestimated sum may also be added to the account from loss by military conscription, again not counting the losses to society through

those forms of poverty which have their primal cause of war.[15]

For nearly a hundred years the armament budget of each civilized nation has been increasing with acceleration and in one way or another it finds ample public sanction for all its extravagances. It bears no logical connection to the need of defense or to any real necessity of the nation. Professor Grant Showerman justly observes that "modern peace is only a near relation of war, of a different sex, but of the same blood." It is the dormant side of war. In the words of Bastiat, "War is an ogre that devours as much when he is asleep as when he is awake." It must be noted, however, that in all lands, voices are being raised against these forms of extravagance, though as yet most of them are "crying in the wilderness." In a few nations only, and in those mainly from reasons of tax exhaustion, has there been any attempt to place a limit on such expenditures.

War Expenditure and National Resources

The "endless caravan of ciphers," which expresses the annual interest on debt and the annual cost of military expenditures, represents the in-

[15] A recent study of Dr. S. Dumas of Paris is interesting in this connection. Dr. Dumas shows that in France, Germany, Denmark and Austria, the death rate among the *people at home* is 12 to 25 per cent greater in time of war than in time of peace. The percentage in Austria for example rose from 2.92 to 3.22 in the war of 1866, in France from 3.28 to 4.06 in the war of 1871. (*The Peace Movement*, Berne, March 30, 1912.)

terest charges yearly on a capitalization of $100,000,000,000. This amount, of which the entire earnings are devoted to past and future war, is just a little less than the estimated value of all the property of the United States ($110,000,000,000). It is nearly double the wealth of Great Britain ($58,200,000,000), practically double that of France ($50,800,000,000), more than double that of Germany ($48,000,000,000) and three times that assigned to Russia ($35,000,000,000). It is equivalent to the total capital of eight states like Italy ($13,000,000,000) or ten like Japan (about $10,000,000,000). Almost the equivalent of the total estimated wealth of Holland ($4,500,000,000) is expended every year for military purposes in times of peace by her ambitious and reckless neighbors. The greater wealth of Spain ($5,400,000,000) or Belgium ($6,800,000,000) would last for a year and a half, while that of Portugal ($2,500,000,000) would be consumed in a single year by the militarism of Europe alone.[16]

The entire yearly earnings of the United States in all wages and salaries amount to $15,363,641,778. This would pay the military bill of our country for thirty years, that of the world for nearly four. The world cost of war for a year consumes the wages (the average being $518) of 8,000,000 American workmen, or of 3,300,000 Americans who work for salaries (av-

[16] Bliss—*Encyclopedia of Social Reform,* p. 1279, 1906.

erage being \$1,188).[17] A similar comparison for Europe would almost double these proportions. The average wealth per capita of the individual man is set down in Europe as \$727, in America as \$1,209. The results of the life work of ten average men will pay for about one minute of the military expenditure of the world. Now if all this is truly necessary to the peace and well-being of the world there is not a word to be said against it. But a matter of such gigantic importance should be justified by very careful study and very complete evidence.

If all civilized nations could be placed on a peace footing, it would be a comparatively easy matter to pay off the national debts. The savings thus achieved would make a new world, in which poverty need not exist as a result of external social or economic conditions, but solely from causes inherent in the individual.

War Debt as a Blessing

A hundred years ago it was a favorite saying that "a national debt is a national blessing." In this view of the case lies a double fallacy. On the one hand national borrowing tends constantly to transfer savings from the common man to the money lenders. On the other, the money borrowed is used for temporary and non-productive purposes. When the "navies melt away," the

17 Estimates based on the Reports of the United States Census Bureau for 1910.

money and the effort expended on them vanish, leaving the nation so much the poorer, as the same money and the same effort might all have been turned into productive channels. If the government presented its bonds gratis to the money lenders, or if bonds or money were stolen outright, the nation as a whole would be none the poorer. It would be a transfer of wealth, whether honestly or not, within the confines of the nation. As matters stand, in the words of Professor Johnson:

"The money raised through the bond issues serves as an instrument for taking men who would otherwise be engaged in productive labor away from their tasks and setting them at the useless occupation of dawdling around the barracks. It takes capital away from manufactures and transportation and embodies it in warships that fifteen years hence will at best serve as targets for still more formidable warships at artillery practice. To ignore the waste of war debt is to elevate military expenditures in the economic sense to the rank of graft and common theft—practices which transfer wealth, but do not destroy it. Militarism steals our wealth and wantonly burns it up or sinks it in a bottomless sea. All that is saved is the financiers' commission and the armourers' profit."

Moreover, the idea that because its bonds are held within its own boundaries, a nation is none the poorer for its debt, involves further fallacy.

There are "empires within empires." For example, though the Rothschilds in London are recorded as English, those in Paris as French, those in Berlin as German, this great house never belonged to any country. It existed and still exists for itself alone. So with all other great syndicates of finance. These groups cannot serve two masters equally. They naturally serve themselves first.

Whether the gold paid out in interest leaves the country or not has no significance, so long as it leaves the purses of the taxpayers. If it leaves the country, something equivalent has been returned. All devices of the spendthrift nation, the indirect tax, the deferred payment, the government monopoly, tend to divert money from the pockets of the common man to the vaults of the financiers. A million little streams of coin unite to form a great river of gold, controlled by the masters of finance. The common man may waste his money. The financier knows how to make it work. We hear of great investments in foreign lands on the part of the chief nations of Europe. These investments do not (outside of France [18] at least) represent the people's savings. They represent primarily usury on war loans, profits on armament and on armament loans, voted by parliaments in excess of fear or excess of "patriotism."

[18] France is a country of small investors. The influence they individually cannot wield is exercised by an oligarchy of great financial establishments, Le Crédit Lyonnais, La Société Générale, and the like.—(Albert L. Guérard.)

The common man, the farmer and the workman have no stake in these international investments.

The Cost of Living

Throughout the civilized world for the last fifteen years (since 1897) there has been a steady increase in the cost of living, a steady fall in the purchasing power of gold, not compensated by a corresponding increase in wages or salaries. That this is not due to local conditions or local legislation is evident, for it affects all nations about equally. It is felt in the same way in provincial Austria and in provincial Japan, as I have personally observed. To this rise in cost, there are, no doubt, many contributing causes, most of which I need not discuss here. It is, however certain that by far the most important of these arises from the world-wide increase in taxation due to the immense increase of the cost of war and war preparation. Taxes, the world over, bear more and more heavily on the middle men. Their margin of profit must be increased at the expense of others. The producer at one end of the series and the consumer at the other bear the increased burden. The final incidence of taxation falls on that social group which has least power to raise its prices, least force to throw off its burdens on others.

In the thoughtful report of the Massachusetts Commission on the Cost of Living in 1910, the commissioners find the most important element to

be "Militarism, with its incidents of war and waste and its consequences in taxation."

"The three great wars of the last decade and a half—the British-Boer, the Spanish-American, and the Russo-Japanese—took millions of men out of the productive activities of our civilization into the wasteful activities of warfare, diverted the energies of other millions from useful industry in shop and mill and farm, and transferred their skill and labor to the production of war equipment, material, food and supplies for the armies in the field. This diversion of labor and capital from productive industry to waste and destruction, with the accompanying diminution of the necessaries of life and an inability to supply the world's demands, inevitably resulted in an advance of the prices of the commodities of common consumption.

"In addition to these conditions, and incidental to them, the mania for militarism leads nations to plunge into debt in order to create and maintain armies that may never fight and navies that may never fire a hostile shot. This mania has piled up huge financial burdens in England, Germany, France, and other foreign countries, for meeting which the best energies of their statesmen are diverted to devise new methods of taxation. In the United States, as in Europe, the exactions of militarism and its burdens of debt are . . . prime factors in the economic waste that has produced high prices. This commission does

not care to discuss the philosophy of militarism. It simply desires to show that war in all its phases is one of the most serious influences in producing present high prices."

One result of the increasing cost of living and the narrowing margin of the wage worker, the world over, is to raise "Social Unrest" to the danger point. Bread riots, tax riots, virulent strikes, "sabotage," "anti-patriotism" and to a large extent Anarchism and Socialism arise in reaction against the oppression of debt and waste. The final result of the upheaval of social forces no one can prophesy.

In a recent German cartoon, the prime minister and the minister of foreign affairs are pictured as watching, from a balcony, the gathering of a crowd of people in a public square. "Sie schreien Moroko! Moroko!" (They cry Morocco! Morocco!) says the prime minister, whose thoughts are on matters of diplomacy. "No," says the other, "Sie schreien Brod! Brod!" (They cry bread, bread!)

For the thoughts of the people were not on Imperial Extension in Africa, "the Mirage of the Map." They were interested in their own immediate affairs, the prospect of escaping starvation.

"To the vast majority of 250,000,000 people, it does not matter two straws whether Morocco or some vague African swamp near the equator is administered by German, French, Italian or Turkish officials." (Norman Angell.)

IV. THE CONTROL OF NATIONS

The financial affairs of Europe, and these include all questions of war and peace, have passed into the control of the money-lenders.

The control of a railway system does not necessitate ownership but simply the control of its debt and its needs at critical moments. Just so with nations. It is the need for more borrowings that makes the old loans dominant. In proportion to the bulk of their debts and the acute character of their need for money are they subject to dictation. The ordinary creditors or bondholders have little to say. It is the necessity for further loans which places control in the hands of the financier. This may be exercised quietly as befits the business of the banker, but it is none the less potent and real.

"Dollar Diplomacy"

Barbarous nations have no debt. With the extension of enlightenment, one by one they fall under the control of the Unseen Empire. To a mild form of transition has been recently given the name of "Dollar Diplomacy." The essence of "Dollar Diplomacy" is the conduct of the foreign affairs of a nation in such a way as to favor the financial operations of its bankers.

In general, an international banking company may have three functions. (1) ordinary banking,

that is, furnishing free money to "going concerns" to be used in profitable enterprise. This requires no assistance from the Government and the many thousands of institutions engaged in it ask only justice. (2) The control and promotion of business enterprises. In so far as these are legal, and their activities extend beyond national boundaries, they have the right to claim the assistance of the Consular Service in the same degree with any other industrial undertaking. Large enterprises have no greater right than small ones to governmental assistance. (3) "Pawnbroking" [1] on a large scale, that is, the placing of national loans.

To assist in the adjustment of foreign loans is no recognized part of the administration of a republic. It is claimed in behalf of the new "Dollar Diplomacy" that it assists in bringing to debtor nations "the blessings of peace, prosperity and civilization." But this phrase with its associate, "Spheres of Influence," covers a type of operations from which our nation has, until lately at least, as a matter of principle stood aloof.

"Spheres of Influence;" (Persia)

The method of working up foreign loans known as "Dollar Diplomacy" is relatively simple and modest, depending not on force but on friendly advice and the suasion of opportunity. The conventional European method of arranging such

[1] See page 13, "Banking and 'Pawnbroking'."

affairs is by means of the extension of so-called "Spheres of Influence." The details of an operation of this kind are prophetically given by a Persian journal (*Hablu'l-Matin*, September, 1907 [2]) four years in advance of the actual occurrence.

Referring to the petty revolt of Salaru'd Dawla, a futile aspirant to the throne of Persia, the *Matin* discusses the supposititious comments of the *Times* and the *Standard* on the necessity as expressed by them, that Russia and England should coöperate to destroy the Pretender and to bring peace and prosperity to Persia.

"Since the disturbed districts were nearer to Russian territory, troops should be brought from Russia, but that the expenses of the expedition would be equally borne by the two Powers. There would be a vote in Parliament, followed by a correspondence with St. Petersburg. The troops would arrive. The Salaru would be taken prisoner. The troops would remain for some time in the district, detained by 'restoring order.' The expenses of all these proceedings would be calculated, and would be found to amount to about five million pounds sterling, which would have to be recovered from the Persian treasury (just as in China they demanded the expenses incurred in sending troops and also a fine). Well, the Persian treasury would practically be unable to pay this sum, so it would be found necessary that an official should be appointed on behalf of each of the

[2] As quoted by Browne, *The Persian Revolution,* 1910, p. 181.

two Powers to increase the revenues and supervise expenditure, and that the Russian official should watch over the North of Persia, and the English official over the South. After a while each would report to his Government to the effect that, having in view the destitution of Persia, the revenue could not be increased, and that the payment of this sum was impossible; and that in some way or other, the condition of Persia must be improved, so that her revenues might be enlarged. Persia, they would add, only needed certain necessary reforms to become more prosperous. Roads and means of communication should be improved; railways were needed in certain places; dams must be constructed to increase agriculture; the erection of factories was greatly needed. Finally, after prolonged discussions, it would be agreed that a sum of at least twenty million pounds sterling must be lent conjointly by the two Powers, of which sum part should be spent on irrigation, part on roads, part on mines, part for administrative purposes, and so on, and that with the remaining two millions a Bank should be established. The Persian Government would, under the circumstances, be compelled to submit to these conditions and sign the required bond, comforted by the assurance that the conditions were very light and easy, and comprised no more than ten clauses, that the loan would cause Persia to blossom like a garden of roses; and that her revenues would increase tenfold!

"The terms of the new loan would comprise at least two clauses, the ratification of which would close forever the charter of our independence. . . .

"One of these conditions would be that the offi-

cials in control of all the financial departments of the Government must be appointed by the two Powers, and that they in turn must appoint the minor officials. These would assume control over all the frontier districts, possibly over the interior also, and would impose a complete check on the functions of the home officials. We need not remind our readers how much one single Belgian official, on obtaining complete control of the Persian Customs, increased the influence of foreigners, or how he caused Persian employees to be ignored and humiliated, and this notwithstanding the fact that we were able to dismiss him at any moment we pleased, and that he had no sort of independent authority in our country. Whoever has examined the new Customs Tariff (drawn up by him) knows of what treason to our country this ungrateful wretch was guilty, how he increased Russian influence, and how he behaved toward the Persians. Hence it will be evident how the Russian and English officials, enjoying complete authority and unrestricted power, and representing Persia's creditors are likely to conduct themselves. . . . Moreover, since the borrowed capital will be under their own control, they will employ it in such a way that most of it will revert to their own countries.

"Another condition will be that all concessions granted by Persia, whether internal or external, must be approved, sanctioned and ratified by the two Powers. Accordingly a Persian subject will neither be able to obtain a concession for the manufacture of paper nor to set up a factory, since the granting of all such concessions will be in the hands of the above-mentioned functionaries, who, in one way or another, will prefer their compatriots to us, so that all com-

mercial undertakings will pass into the hands of Russian and English merchants.

"Another condition will be that these officials shall receive their salaries from Persia, who will recognize their claims and rights, and, in return for their services to their Governments, they will receive a yearly payment in cash from the Persian treasury. . . .

"Another condition will be that all the material wealth of Persia must be handed over to guarantee the debt. This stipulation will include the mines, coasts, customs, ports, telegraphs and revenues, and since the debt must be paid out of these sources of wealth, and the Persians do not know how to manage them or put them to profitable use, therefore officials appointed by the two Powers must superintend them and take such steps as may be required to render them productive. The Persian Ministers must therefore be subordinated to these foreign officials, whose commands and prohibitions they will not have the slightest right to disregard."

The late Amir of Afghanistan, Abdur Rahman, is quoted as saying that "Russia is like the elephant who examines a spot thoroughly before he places his foot down upon it and when once he places his weight there, there is no going back and no taking another step in a hurry until he has put his whole weight on the first foot and smashed everything that lies under it."

In general the course of military pacification lies along the lines above indicated. The presence of alien soldiers breeds chronic disorder. To re-

lieve this requires more troops and more money and the end of it all is the submergence by debt of the "pacified" nation.

The present condition of Persia is thus summed up in the following paragraph:

"Persia's unexpectedly setting out to pay off the mortgage and rebuild her house so alarmed the covetous mortgagees that they did not hesitate at highway robbery to keep the redemption money from being paid." (*The Nation*, N. Y., Jan. 25, 1911.)

"The allies are demanding heavy money 'indemnities' which they will take good care to make large enough to preclude the possibility of their payment without recourse to a large foreign loan. This in turn will be furnished the unfortunate Persians only on such conditions as will effectually mortgage for years to come as many of their resources as can be found not already assigned to foreign syndicates. The Powers will do their best to leave no money for future Shusters to collect. The Persians, poor wretches, cannot be allowed to govern themselves well or ill; they are misguided enough to live in a country possessed of strategical importance, and they must take the consequences." [3]

"Continuity of Foreign Policy"

It is understood that the recent policy of Great

[3] Professor Roland G. Usher—"The Significance of the Persian Question"—*Atlantic Monthly*, March, 1912.

Britain, as exemplified in various episodes in Asia and Africa, characterized by alternate cringing and bluster, by scrupulous justice and studied injustice, by artificial "ententes" and artificial enmities, is dominated by the great "law of continuity of foreign policy." In other words a great and enlightened nation is forced to live down to its worst lapses in international courtesy and moral dignity. The ingenious Mr. Chesterton remarks: "There is very little doubt as to our national vice. An acute observer of Russia says that nation lacks the cement of hypocrisy. We do not."

Money and the Morocco Affair

As already suggested, the influence of the "Unseen Empire" now makes for peace and for solvency. It controls and creates the credit of Europe. It will not connive at its own injury. It is said that the Bank of England has a "psychological reserve," which guarantees its solvency in every crisis. The pride of England is involved in its maintenance. To default its pledges would mean the collapse of credit. The great bankers hold similar relations to the Credit of Europe. War is a disease which spreads to every function of the nation. While the bankers might make large temporary gains through reckless discounts, in the long run they would be the losers through the disturbances of international war. They guard the solvency of the world. An il-

lustration of their influence is seen in the late Morocco affair.

The following account of the Morocco transactions is condensed and slightly modified from an article by Francis Delaisi.[4]

"At the end of August, 1911, the sharp crisis between England and Germany was over. The understanding between Paris and London made war impossible. There was no other course but to make an amiable settlement as smoothly as possible. The French demand was for (1) a political protectorate over Morocco, that is, the right to have her soldiers killed there and to spend millions in order to maintain regular administration, and (2) the monopoly of loans for the public works of Maghreb. This would give the French business men the compensation for these costs.

"The German offer was that of economic equality; whereby the Germans would participate equally in all gains, leaving France the glory of possession and the expense. Finally, in exchange for this, they asked all of French Gabon and the middle Congo. According to the familiar illustration of Frederick II, they would squeeze the orange, taking the juice, leaving the rind to France. . . .

"Germany, thanks to her marvelous commercial development, is growing rapidly rich. Though not poor, she has not yet the great capitalism of the old nations like England and France. In her industries she needs large sums for short periods, and for these she goes to the most abundant market, that of Paris. . . .

[4] "Financiers contre Diplomats," *Echo de l'Ouest.*

"The French Banks recognize two types of international loan; consolidated loans on long periods and advances at higher rates on short terms ("titres en pension").

"With a loan of 25, 30 or 50 years the creditor can in no case try to collect before the agreed date. Thus, for example, if it pleased our great friend the Czar, the morning after one of the loans of 1200 millions of francs which he makes from us regularly every four years, to abandon us and to attach himself to Germany, we should have nothing to say. He could, in need, declare war, and use to fight us our own millions. If we wished to constrain him by force of arms to return our money, we should only ruin our security. Each one of our victories would cut down the value of our bond.

"The German demand for economic reasons is for short loans, to push on the work of manufacture and commerce. Ordinarily these loans are granted and renewed as needed. But if trouble arises between the republic and the Kaiser our financiers have only to give the sign and our money returns to our own treasure vaults. Such an action would have grave consequences to Germany. Let us suppose that the French bankers recall suddenly the 700 or 800 millions advanced to the German Bank. The rate of discount would rise at once from 4 per cent to 5, 6 or even 7 per cent. With this, current industrial profits would be swept away.

"This is a powerful weapon which M. Dorizon bore with him on the tenth of August, 1911, when he carried in his pockets a counter-project to that of M. Kiderlen-Waechter. If the imperial chancellor had held out, the banker could let loose on Germany a

fearful crisis in finance. The machinery of the "titres en pension," the short time loan, which has been figured as a sort of betrayal of our own interests, is thus transformed into a terrible weapon as against our adversaries.

"We shall see how our financiers have known how to use it, and how by its use, they have conquered the arrogant diplomacy of our neighbors."

Their own interests not being in jeopardy we may naturally expect the money-lenders to be indifferent to questions of international morals, and ready to finance either side alike. The safe limit of international loans being reached, as in the Russo-Japanese war, the dawn of peace is not far distant, however widely variant the claims of the contending parties.

Money and the Tripoli Affair

In the present war between Turkey and Italy, it is recognized that both nations have practically reached the limit of tax exhaustion. It is claimed that the "Unseen Umpire" has declined to make any further loans to Italy. The last loan granted to Turkey involved a heavy bonus, something like 17% in advance. It is further understood that in the diplomacy of Europe, Italy has leave to take her share in Northern Africa at her convenience. Again, it is believed that Tripoli would be acceptable as a basis of further loans. Presumably, in the end, Italy will receive Tripoli, paying a certain sum in exchange to the creditors

of Turkey. All this is hypothetical and uncertain, but it forms a working theory of the reasons why this war has been permitted by those who are in position to prevent it. It is claimed that the present desultory warfare is being carried on by means of the earnings of Italian emigrants, deposited in the Banca di Italia.

Cost of a Small Modern War

The question as to whether it will all pay seems not to have had due consideration from the statesmen of Italy. The "mirage of the map" is in this region especially elusive. While actively prosecuted, the war is estimated to have cost Italy from four hundred thousand to a million dollars per day. These sums may serve to expel the Turkish garrisons from the coast cities of Tripoli, but the conquest of the Arab tribes of the desert is another matter. Authorities disagree as to the amount of Tripoli's foreign trade. Taking the highest estimate as to its exports and admitting a generous per cent. of this as profit, Italy's gain from a year's peaceful occupation of Tripoli under present economic conditions would scarcely pay for a day of war. And even such pittance would go, not to Italy, but to those men, Turks, Jews, Italians and French, who might chance to control the export trade. Doubtless under better government trade would increase. Possibly under favorable conditions the profits on a year's trade might be made to cover a whole month's

military expenses. But all Italy will get is the "mirage of the map" and even this seems to waver a bit. Her previous melancholy experience in the invasion of Abyssinia will be repeated in Tripoli, and this whether at the end she finds herself victorious or not. Whatever the moral questions involved, it is certainly bad economics for a nation to indulge in a raid the cost of which far exceeds the booty.

Cost of Armageddon

If desultory warfare between a second-rate power (really become third-rate through burden of debt) and a third-rate power (now become fourth-rate) is so expensive, what of "God's Test" of the nations lightly prophesied by certain militarists? [5]

[5] "At the present epoch in the world's history Mr. Carnegie might just as well have created a trust for the abolition of death. . . .

"The real Court, the only Court in which this case (Japanese immigration) can and will be tried is the Court of God, which is War. The Twentieth Century will see that trial, and in the issue, which may be long in the balance, whichever people shall have in it the greater soul of righteousness will be the victor. . . .

"Never was national and racial feeling stronger on earth than now. Never was the preparation of war so tremendous and so sustained. Never was striking power so swift and so terribly formidable. What is manifest now is that the Anglo-Saxon world, with all its appurtenant provinces and states, is in the most direct danger of overthrow final and complete, owing to the decay of its military virtue and of the noble qualities upon which all military virtue is built. . . . The voice of every God-

I quote below from a Paris correspondent of the American Associated Press (*San Francisco Chronicle*, January 2, 1912). Whether the details are wholly correct or not is a matter of minor importance.

"Europe is preparing for war. It has battalions, ships, howitzers, steel-clad automobiles, aeroplanes for dropping bombs and bombs for making aeroplanes drop; swords to cut you to pieces and surgeons to sew you up again. In fact, the war machine is faultless. But who has got the fuel to set it working? Who will plump down the cash for war between England and Germany, and for the resulting Armageddon between the Triple Alliance and the Triple Entente?"

"Is Herr Bebel right when he says the coming Franco-German war will cost $750,000,000 a month or $9,000,000,000 a year? If England, Austria, Italy and Russia join in, may it be assumed that the war will cost in proportion—that is, $27,000,000,000 for the first year, not to mention the second? And who will pay the $27,000,000,000?

"Europe's brilliant statesmen waste no time on this

fearing man should be raised . . . to revive that dying military spirit which God gave to our race that it might accomplish His will on earth. . . .

"The Shadow of Conflict and of displacement greater than any which mankind has known since Attila and his Huns were stayed at Chalons is visibly impending over the world. Almost can the ear of imagination hear the gathering of the legions for the fiery trial of peoples, a sound vast as the trumpet of the Lord of Hosts."—(Harold F. Wyatt, "God's Test by War"; *Nineteenth Century*, April, 1911.)

insignificant problem. They are far too busy brewing those glorious wars—a business easier far than carrying them gloriously on. But frivolous, unpolitical people—financiers, economists, statisticians, traders—would like to know who will pay?

"First, they ask themselves what will a war cost? No man knows. Herr Bebel's estimate is guesswork and probably exaggeration. Italy's little war with Turkey is costing $400,000 a day, allowing for a mere 60,000 fighting men. Professor Viviani of Rome says that if only 80,000 men are sent it will cost 1,000,000,000 lire or $200,000,000. Since this estimate was made it appears Italy will have to send 120,000 men; and if she marches into the interior, still more. The Boer war, in which England's army averaged 200,000, cost $1,055,000,000 in two and a half years. The Franco-German war, which lasted only 190 days, cost Germany $450,000,000 for an average fighting force of 1,250,000 men. The war in the Far East cost Japan $650,000,000 and Russia $723,000,000, not counting lost ships. Only toward the end had either side anything like a million men in the field.

"The coming Armageddon will cost infinitely more than these, because the armies in the field will be bigger and because food, clothes, arms and ammunition every year cost more, and more of them will be wasted. Moderate estimates are that a war lasting a year will cost France, Germany and England each about $2,300,000,000. Russia's yearly bill would be $2,800,000,000 and Italy's and Austria's about $1,400,000,000.

"To meet such demands and to prevent universal panic no single European state has made proper pro-

vision. Europe's war chests consist mainly of Europe's capacity for borrowing. Germany alone has an 'Imperial war treasure,' which has captivated the European imagination, but in reality is ridiculously small. This is the Reichskriegsschatz, which lies in the Julius tower of Spandau citadel, guarded by triple steel doors, 'simultaneous keys' which are held by different individuals, and a dozen sentries. It amounts to a beggarly $30,000,000, all in coined ten and twenty mark pieces, kept in boxes, each of which contains $25,000. Against it are issued $30,000,000 in imperial treasury notes, so that no interest is lost.

"Germany's real war asset is her state railroad system. No country has such a splendid asset. The Prussian railroads, which cost $2,250,000,000, are now valued at $4,500,000,000. They pay an average of 7 per cent on the invested capital. The result is that only $1,500,000,000 railroad debt is outstanding, so that the state owns railroad property which has a sale or mortgage value of $3,000,000,000 clear of debt. The South German state railroads are not so remunerative, but they could be mortgaged for a considerable sum, and optimists have declared that on her state property Germany, if need be, could secure loans totalling approximately $4,300,000,000.

"Russia is the only other state that can talk of realizable assets in terms of hundreds of millions of dollars. First comes the $495,000,000 gold reserve which lies in the cellars of the Bank of State. Against it are issued $610,000,000 in credit notes, and about $400,000,000 more in credit notes could be issued in war time without exceeding the legal limit. That is one asset. The 29,000 miles of state rail-

ways cannot be counted an asset because they show an annual deficit of \$10,000,000. The greatest Russian asset is the 'State Vodka Monopoly,' which was started as an experiment by Witte in 1895, and is now in force all over the empire. Under it the state is the sole manufacturer, wholesale and retail liquor dealer. The monopoly has tended finally to demoralize the muzhik, but as a financial measure, it has been a brilliant success. It yields an annual profit of about \$225,000,000 and has a capital sale value of about \$4,000,000,000.

"At first sight, England and France, Europe's richest two states, are far worse off than relatively poor Germany and very poor Russia. They have practically no assets. France has only the tobacco, match and powder monopolies, and England has her posts, but neither can produce an asset like the Prussian state railroads and the Russian drink monopoly. Yet the national credit of England and France is more valuable than these two put together. Considering the far lower interest, English Government stocks are quoted much higher than German. England's credit, even during war, is better than Germany's during peace. In March, 1900, England issued at 98½ a 2¾ per cent Boer war loan of \$150,000,000. It was subscribed eleven times over. In the same month Bavaria appealed for a mere \$10,500,000. For this she had to consent to interest at 3½ per cent and an issue price of 93½. The late Sir Robert Giffen said that on emergency England could appeal for \$500,000,000 at 8 in the morning and have it at 8 o'clock at night. The credit of England and France is based on their national wealth and accumulations of capital.

"When war breaks out this laying hands on savings will be the immediate unromantic occupation of the brilliant statesmen who have brought it about. The first steps will be to transform the state banks into war-banks, and to proclaim *le cours forcé*—that is, that Government issues must be accepted at their nominal values. In Germany in time of peace, the issues of the 'Reichsbank' may be rejected and gold demanded. A second war measure will be to suspend the periodical publication of the level of the state banks' gold stock. This also was done by France in 1870. The economist Stroell holds that Governments will further finance war by issuing 'forced paper,' without any gold backing, thus keeping their gold reserves intact. The only security for such paper would be the Government's credit. But England and France are ahead in both the conditions of their stock exchanges and of their savings banks. The German savings banks are weak because their money is largely invested on mortgage and cannot be quickly realized in case of panic.

"Looked at from the money point of view, the European Armageddon is not at all the obvious, simple thing which people imagine. It is a complex, menacing question, full of disagreeable surprises and treacherous pitfalls, and it promises the ruin of at least some, if not all, the states which rush into it lightheartedly. But Europe's brilliant diplomatists do not worry themselves about such contingencies. What with diplomatists, statesmen, shipbuilders and gun makers—not to mention The Hague arbitration judges and the bloody-minded peace societies—all trying to bring about a war, a war is some day inevitable. But who will pay the piper?"

Nevertheless, however big the war cloud, the storm will pass over. We may be sure, to use Mr. Powell's words, that "there will be no war until the real rulers of Europe from their strongholds in Lombard Street and the Rue Quatre-Septembre, in the Burgstrasse and the Schottenring themselves tell the fighters to fight."

Interest of "High Finance"

In the *London Nation* (January 21, 1912) occurs the following:

"Fortunately it is to the interest of *la haute finance* that France and Germany should live in peace. Indeed the last thing that the financiers desire is a European war. The advantages of adventures in Asia or Africa may lead them to endanger the peace of Europe, but they always think that they can prevent the danger of war from becoming a reality and, in fact, they did so in the recent crisis. For the present they will be content, no doubt, with the advantages already secured, and we are not likely to have a repetition of the Moroccan venture."

The present relation of international finance to international war is thus convincingly presented by the *New York Tribune:*

"On more than one occasion in recent years we have been able to discern unmistakable indications of the sway of . . . 'the unseen empire of finance' in international affairs, and particularly in the avert-

ing of wars and in the promotion of friendly relations among the powers.

"From one theoretical point of view that is, of course, to be much deplored and disapproved. The nominal rulers of a land should be also its actual rulers. The exercise of control by an irresponsible 'power behind the throne' is too susceptible of abuse to be approved. Yet in practice, by the very showing of its critics, this system has had certain good results. It has consistently made for peace. That is indisputable. Nobody has ever heard a hint that the Rothschilds were trying to bring on a war. Their influence has always been for peace, for the very practical reason that in peace lie their security and their profits. There is abundant reason for the belief that on several occasions this money power has been the chief barrier between Europe and a devastating war. To that extent, therefore, the rule of this 'unseen empire' must be regarded with approval.

"There is, however, a paradox involved in the case, in the circumstance that the moneys lent by these capitalists to the governments are used largely for military purposes. They are, in fact, potential war loans, made by those who desire the keeping of the peace. Capitalists do not hesitate to advance money for naval construction or for army enlargement. Indeed, there have at times been suspicions that they were using their influence in favor of the making of large appropriations for such purposes, which would necessitate the issuing of loans. And then, after the loans are made, they exert their influence to prevent the fulfilment of their intended uses. It is an interesting subject of speculation whether the influence of the provisions which are made by the loans, or of the

makers of the loans, will in the end prove the more powerful."

Norman Angell[6] shows the growing interdependence of nations to be such that even a policy like Bismarck's could not bring on war between Germany and France.

"Where Germany could have 'bled France white' with a certain satisfaction without any immediate damage being involved to his own country, Herr von Kiderlen-Wächter (I am told to his surprise), learned that to 'bleed white' this relatively feeble France of 1911, would be to plunge this great and powerful Germany into the direst economic distress. . . . The very threat of it was enough. . . . I could trace for you a really humorous chart establishing the direct relationship between the vigor of German foreign policy and the figures of German commercial insolvency."

The experience of the world shows that no nation can have at once a great army, a great navy, a great debt, a vigorous foreign policy and the prosperity of its citizens. Two of these it may have and sometimes three, but never all five at once.

In another article,[7] the writer above quoted refers to the solidarity of industrial and commercial interests as illustrated by the international rela-

[6] *Public Opinion,* February 2, 1912.
[7] *Public Opinion,* January 26, 1912.

tions of the bankers of the world. He asserts that "banking all unconsciously is bringing peace to the world by making nations financially interdependent: that the material side of wealth as represented by banking makes for and not against a better human society and the higher welfare of the race; that the world's granary will have enough to spare for all mankind when men and women work together for mutual good."

War and Modern Banking

Some time since Lord Rosebery, noting the movement in society from personal and arbitrary rule, said that "royalty is no longer a political but a social function." A change similar in character has modified the nature of banking. Banking is becoming not a political but a social function. Many of the misfortunes of Europe have been ascribed to the early alliances between politics and banking.[8] That alliance is being dissolved. Banking, as well as government, is under the growing influence of democracy and cosmopolitanism. While "pawnbroking" on a large scale still concerns itself with affairs of derelict, inchoate or helpless nations, and is therefore still

[8] For the most part "it is not the banker who wants to interfere with politics. It is the politician who wants to interfere with banking. All that the banker generally asks of politics is to be left alone." . . . . "Separate even the most powerful of these 'sinister figures' (of the international financier) from the interests or the economic forces of which at the moment he may be the representative and he is reduced to practical impotence." (Norman Angell.)

allied with politics, the spread of common banking causes it in a way to become the nervous system of society, its protection against economic harm. Norman Angell [9] has clearly shown that in public affairs the banker is the first to feel the symptoms of disorder. War is sickness in the economic as well as in the social organism, and the influence of sound banking is everywhere and automatically opposed to it. To the modern banker, as to Benjamin Franklin, "there never was a good war nor a bad peace." "Destruction of capital, in the nature of things, never appeals to a banker." [10]

The influence of finance now lies, not in spectacular provision of great loans, but in "the unnoticed impersonal forces which the ordinary weekday humdrum work of banking has called into existence, the cumulative outcome of those numberless every-day operations that take place almost completely outside the control of Governments and of financiers, often unknown to them, often in

[9] "If we can imagine an animal that did not feel hunger or cold or the bad taste of poisons, it would very soon be wiped out. It has nothing to guide it in its adaptation to its environment, none of the acute promptings which result in placing it in the most favorable conditions to allow the unconscious and uncontrollable processes to be carried on favorably. Now, banking is performing, among other functions, this immense service to the economic and social organism; it is providing it with sensory nerves by which damage to any part or to any function can be felt, and thanks to such feeling, avoided." *(Influence of Banking upon International Relations:* Norman Angell, 1912.)

[10] Isaac N. Seligman.

spite of them, representing forces far too strong and far too elusive for such control, so much a part of the warp and woof of the ordinary life of the world that they are rapidly and surely weaving society into one indissoluble whole." [11]

According to Mr. Isaac N. Seligman,[12] "the Russo-Japanese conflict of 1904-5 was halted in large measure because bankers refused to float further loans at anything like ordinary terms after probably $1,500,000,000 had been wasted in that contest. The interests of commerce have thus put into the hands of international bankers a powerful weapon to use in the interests of conciliation and peace." Mr. James Speyer has proposed that this weapon be formally adopted and consistently used against those nations who may try to disturb the peace of the world by making war, as all war must be made, on borrowed money.

"If by withholding the 'sinews of war' the banker can force a nation to desist from war, he conserves to its people the enormous sums which would have been wasted" (Seligman).

Banking is the democratic phase of what under autocratic rule is pawnbroking. In common with other influences of democracy banking makes for peace. It is a function of peaceful industry as pawnbroking is of war. "A king without money is like a spear without a head."

[11] Norman Angell.

[12] International Banking and International Unity Conciliation Society, No. 50, 1912.

V. SEA POWER

In this chapter is given an account of "Sea Power" in its relation to national affairs. It is the most costly element in the business of government. The very name has in itself a magic which unlocks the strong boxes of the world.

Armament Competition

The present status of "Sea Power" is thus graphically summed up [1] by Professor William I. Hull:

"Each nation argues that it can protect its own peace only or best by increasing its armaments; and accordingly each of the circle of forty-odd nations is feverishly engaged in the edifying task of out-arming, to the best of its abilities, each of the others. Great Britain, assured that her own peace and the peace of the world is threatened by the menace of the Teuton, lays down the keels of two dreadnaughts; Germany, perceiving the portentous shadow of the advancing Briton, lays down the keels of two super-dreadnaughts. This gives to Great Britain a realizing sense of the inadequacy of her twenty-eight miles of warships, and in order to avoid another panic such as the German super-dreadnaughts caused her, she increases her per capita naval expenditures within ten years by 43 per cent.; Germany 'goes her several better,' and increases her per capita naval expenditures

[1] The World's Two Vicious Circles: *Advocate of Peace*, December, 1911.

within ten years by 119 per cent. Some American 'statesmen dream of the menace of Germany in South America or Japan upon the Pacific, and the United States, frightened by such nightmares, increases its per capita naval expenditures within ten years by 64 per cent. Japan, emulating its Occidental school teachers in their fallacious logic, and postulating the impossibility of having too much of a good thing, increases its per capita naval expenditures within ten years by 137 per cent. The other four 'great Powers,' caught up in the same frenzy of fallacious logic and futile competition, convert their national resources into dreadnaughts, and all eight together expend upon their navies within ten years the almost unimaginable sum of $5,600,000,000! [2]

"Thus the vicious circle is formed; the small members of the family of nations join in the frenzied competition for big, bigger, biggest armaments, and, like the serpents of an African jungle, each struggles and strains to raise its head high above the others. But how much like a will-o'-the-wisp is the peace based upon such a chain of reasoning is shown by the continually precarious and fragile character of that peace, while above it broods the shadow of a menacing Armageddon unrivalled in history or prophecy."

Purposes of Sea Power

The end of almost half the military expenditures of the world is to develop "sea power." For this, nearly every sea-faring nation, great or small, spends more than the cost of all its civil

[2] Figures from the British Admiralty's "White Paper" of October, 1911.

equipment, lavish as this sometimes is. And yet the function of "sea power" is most vaguely understood by the people who pay for it. Seven reasons are variously given for its maintenance:

(1) National defense, (2) maintenance of peace, (3) protection of commerce from pirates and belligerents, (4) circumventing of other nations, (5) protection or subjugation of alien dependencies, (6) ceremonial purposes, and (7) "Control of the Sea."

(1) So far as the United States is concerned, the first item, that of national defense, may be regarded as negligible, as fortified towns with their torpedo boats, mines and 16-inch shore-guns are impregnable to battle-ships, unfortified towns under the laws of war are now immune from bombardment, and no modern army can subsist in a hostile land without a tremendous train in the way of supplies. Since Napoleon's time, no army has lived on the enemy's country. Neither does the battle-ship any longer ravage the coast, burning villages or robbing farmers. It is far too costly a tool for such use. A battle-ship has equipment for about an hour's warfare with an enemy of its own grade. Broadly speaking, one may say that after an hour's actual fighting those ships which are not victorious are captured, sunk or fled. The great sea-fight off Tsushima was settled in twenty minutes.

The defense of England on the other hand, depends primarily on her navy. She cannot feed

herself and looks to other nations for bread. But her navy is now swollen far beyond reason and so becomes in a degree a menace to world peace.

If we consider naval expenditures as "insurance" whether on land or sea, the rates are far too high. The "risk," such as it is, applies only to seaboard property and to very little of that. For the most part such insurance is wholly needless for reasons elsewhere given.[3] Further it fails of its purpose because it does not reduce the risk. The more numerous the engines of war, the greater the chance of collision, except as held in check by the operations of bankers.

(2) The second item, the maintenance of peace, we may neglect as a figure of speech. "Sea power" makes for peace through awe or through exhaustion. The first of these keeps the little nations quiet. The second places war out of reach of the large ones, a matter which is discussed further on.[4]

(3) The third item, the protection of commerce, has but limited range of value. In all normal conditions commerce is wholly independent of naval operations. The trade of great nations is, for the most part, not with their dependencies but with their equals. The commerce of Norway and Holland, without sea power, is greater per capita and greater in relation to national wealth

[3] See page 159.

[4] See page 98.

than that of Great Britain or Germany. Even the trade of Switzerland stands in proportion, it is claimed, above that of the great sea-faring nations. Exploitation is not trade. In the control of colonies, the idea of spoliation must be given up before commerce begins.

(4) The fourth reason, the circumventing of other nations, involves the old fallacy that each nation is an individual, mean and grasping, ready at any instant to pounce on its unprepared neighbors. This Pierre Loti calls the "hyena" idea. Nations, however, are groups composed of men each intent on his own affairs, and wholly opposed to collective action which shall interfere with these. The "hyena" spirit sometimes appears in antiquated diplomacy, in reckless journalism or in the belated utterances of case-hardened war-makers,[5] (who have never fought a battle), but not often in the actual interrelations of modern states. Commerce in civilized nations is a mutual affair, and in every cargo that crosses the ocean many nationalities have a stake. The moneyed interests of every civilized nation are to-day bound up with those of every other.

(5) As regards the fifth reason, the protection or subjugation of alien dependencies, the navy can do little more than to serve as a convoy for transport ships. In time of peace this service is scarcely needed. To accomplish it in any

[5] For example, see General von Bernhardi's *Deutschland und der nächste Krieg*.

event would require neither large vessels nor many of them. If it means a great fleet, the game is not worth the candle. Some of England's wisest have doubted whether it is ever worth while. Others have called it the "White Man's Burden."

We may note further that actual war is conducted in the main by armies. Ships cannot fight armies, even as "a herd of whales cannot fight a herd of elephants."

(6) The sixth reason, that of ceremonial needs, is one not often emphasized, but it is obvious that great navies exist largely for giant decoration. It is said that 167 vessels took part in the late Coronation of King George V. Only one of these had seen service, and that was of little importance.

The same spirit which led the Emperor of Germany to add to his honors the title of "Admiral of the Atlantic" is involved in every movement for naval extension. The love of national display, peculiar to no one nation, finds satisfaction in naval parades. In America our biggest ships, however well constructed and however skilfully managed, are valued largely by the people as a decoration. We must not forget, however, that engines of destruction were built for use; they tend always to achieve their normal function, they are at once costly and dangerous beyond all other toys.

Some time since, Norway celebrated at Bergen

the anniversary of the birthday of the great naturalist, Michel Sars. From the Imperial University of Tokyo came an eminent Japanese scholar to do honor to his colleague. At Bergen lay the little Norwegian fleet of battle-ships. Some one remarked that, compared with the fine navy of Japan, the guest must look down on this paltry fleet.

"No," said he, "the reproach of Japan is that she has a great navy with no Michel Sars whose birthday she can celebrate."

(7) The seventh reason assigned, the "Control of the Sea," rests on an outworn anachronism. In 1493 Pope Alexander VI divided the ocean between the two great sea-faring nations of the day, Portugal taking control of all east of about the fiftieth meridian, Spain of all that lay to the west. This gave to Portugal Africa and Brazil and to Spain the rest of the tropical world. But in our day no nations control the sea. All governmental authority stops at the three-mile limit. The open sea is a highway for all peoples alike. In time of war, under the present antiquated code, navigation may be a bit dangerous to merchant vessels of belligerent nations. To-day, however, war never lasts long. It is now beyond any nation to carry it on for more than a few months with a foe of equal resources. A million dollars a day is a moderate cost for a moderate war. Large wars can be had at a proportionate rate. But the lanes of traffic are

soon open again and control of the sea no longer exists.

To say that the United States must fight Japan for "Control of the Sea," as some of our armament promoters have claimed, is the height of absurdity. There is nothing to fight about, and the fight over, nothing would be settled. There is room for a thousand merchant ships on the Pacific where one now sails. If by control of the sea we mean the fact that one nation has more merchant ships than any other or even all others, very well. This is not a matter of war and has no relation to war-ships. The Pacific may, as has often been said, be "the scene of the great deeds of the twentieth century," but these will be deeds of peace and constructive effort. "Sea Power" will then disappear as a nightmare of history.

Abolition of Legalized Piracy

Moreover one of the next acts of The Hague Conference is almost certain to be that of the neutralization of all merchant and passenger vessels. When this comes about piracy will no longer be one of the great evils of war.

Under the laws of war as accepted in 1899 after the first Hague Conference, private property on land, unless used for war purposes, is immune from seizure or destruction. Thus far, under these laws private property at sea may be seized by the crews of hostile vessels and appropriated, as prizes, to their personal benefit. The right to

plunder is supposed to stimulate officers and men to renewed activity. Great Britain has upheld this right, presumably because she has more men to encourage. But she has also most to lose and some of the ablest of British statesmen are now in favor of the neutralization of non-combatants and their property on sea as well as on land.

The old point of view of the English admiralty was expressed in 1861, by Lord John Russell, that England with her superior navy must aim at ruining the commerce of the nations at war with her in the shortest possible time after the outbreak of hostilities, and thus ensure not only her overlordship of the sea but also her supremacy of trade for all times. . . . "It is impossible for other nations to take, lying down, such a perpetual menace. . . . The chief essentials to-day for the maintenance of peace are the general enforcement of the principle of the 'open door' and the general recognition of the inviolability of private property at sea.[6]

While at the beginning of each international war of the last century, a certain number of ships have been seized as prizes there is no evidence that the final results of any conflict have been in the least affected by these preliminary acts of piracy.

Keeping Step

The excessive cost of armament when recognized is usually considered inevitable. A Jap-

[6] Professor Lujo Brentano: Munich, 1912.

anese writer in the "Chuo-Koron" says: "No doubt the war taxes of 160,000,000 yen per year are destroying our country, but the strain of international relations will not allow us to lower our taxes. What we have to do is to strive to increase our natural wealth so that the burden of taxation will not seem so heavy. To reduce Japan's army is impossible, owing to the necessity of looking to the future of China, while to reduce expenditures in the navy is equally impossible. We must do everything to keep the prestige of our glorious navy."

Lafcadio Hearn says that "the Japanese farmers wade knee deep in the mud to produce the rice they cannot eat themselves in order to buy poorer rice and let their Government build battleships to show that Japan has a place among the great Powers."

One of the ablest of Japanese statesmen, and himself opposed to the policy of debt, voices the common feeling of Japan that the chief reason for the development of the Japanese navy lies in the increase of armament of the United States.

This is not that Japan really supposes herself in danger from the United States. It is rather that the larger nations set the fashion. It is feared also that the financial credit of Japan, jealously guarded by her ministry, may suffer if she fails to keep step with her sister nations.

Bad habits are catching. In Argentina it is only a "banal commonplace to observe that the

peace of the world has no better support than naval preparation in every quarter." The journal, *La Argentina*, continues:

"The demand for a naval fleet of Argentina does not imply the possibility of conflict with our neighbors. The ground for this demand is that the rich productions of the country make oceanic navigation indispensable, with the result that Argentina should be in position to make her merchant flag respected.

"Away with weak and idle pacifism and down with all those who oppose tooth and nail the project for a powerful Argentine Navy.

"The arguments stated above point out to the people at large and to the powers that be the importance of a complete reorganization of our Navy, especially as regards the material, size and number of its ships. At the same time we must condemn the attitude taken by our pacifists and the falsity of the position assumed by those who persist in a systematic opposition to all schemes of naval expansion."

The Monroe Doctrine

It is claimed by certain militarists, that "the Monroe Doctrine goes as far as our navy can reach and no farther." In other words, this slogan of the republic rests on force and force alone. If that is the case, it is not worth the cost. If it rests on force and not on right, the sooner it is done away with the better. It may be indeed that it really has no claim on our respect. It may be, also, that our occupation of the Philippines has already repealed the Monroe

Doctrine. For there is a sort of Golden Rule among nations, that no one of them can do what it forbids to its neighbors.

If the Monroe Doctrine has any validity, it is a definable part of international law. It was originally a proclamation against European spoliation in regions geographically allied to the United States. It is fair to others that the doctrine should have a modern formulation by experts in international law. It is a reproach to ourselves that this has never been done. Such a formulation should command the assent and approval of statesmen in all nations. The point at issue is that the nation stands for justice, not for the protection of delinquents.

The Monroe Doctrine would not necessarily forbid transfer of sovereignty even from an American republic to a European empire. It would prohibit its transfer by force of arms. It is not evident that we have any right to go further than this. The belief that we may do so is giving effect to the contrary "Calvo Doctrine" that Latin America can take care of herself. Allied to this is the "Drago Doctrine" that no nation should collect money for its subjects by force of arms.

Unwillingness to Pay

In one of his many discourses on sea-power, Admiral Mahan suggests that the growing unwillingness of the people, the world over, to pay for it may be due to their "degeneration." "De-

generation," as thus used, is a word without meaning. The only "national degeneration" known to history is found in the reduction of the average force of the units of which a nation is composed. Such reduction may be due, as a temporary matter, to poverty or to failure in education or to oppression of any sort; or, as a permanent matter, to emigration, to immigration or to war. Emigration in many parts of the world has lowered the average at home by taking away the best. Immigration may lower the average in any region by filling it up with poorer stock, "the beaten men of the beaten races." The general effect of war is to destroy the virile, leaving the commonplace to reproduce their kind. In this sense, war and immigration have each produced a varying degree of "degeneration" in most parts of the world.

But we can find better explanations for the increasing aversion of the people to borrowing more money for more sea power. Their growing sense of tax-oppression on the one hand, their rising intelligence on the other and the increasingly murderous cost of the whole thing seem to furnish adequate reasons. It may be true, as Admiral Mahan indicates, that the growing cost of armament no more than keeps pace with the increase of national wealth. (In other words, it follows "Johnson's law," to which I have several imes referred.) But in this discussion, we may sk whose wealth it is that keeps pace with the growing cost of militarism. A nation's capacity

to pay is not measured by the swollen fortunes of armament-builders nor of those financiers "without a country" who loan the necessary money. In the long run, it is the common man, the ultimate producer and the ultimate consumer, who pays for all. In all waste production and in all waste consumption, the cost falls on the worker in the end.

Sea Power and Poverty

There is no question that the excessive and growing cost of armament is one of the great factors in national poverty. The greater the sea power, the less the nation has for other purposes. It is agreed in Great Britain for example that to strengthen the army is to weaken the navy, the limit of taxation being already so nearly reached. As sea power grows, the nation weakens through loss of reserve power and through the stress of taxation. The weaker the nation, the greater its need of sea power. In this paradox we find a clue to the persistent state of alarm in England, whose sea power fairly balances that of all her rivals taken together. She fears Germany on the one hand, her own unhappy proletariat on the other, and no accession of sea power can protect her from internal discontent. England is rich, if you look at her from above. The great dukes got her land—for nothing and free from taxes—in the early "merry" days when a county might be given to a royal favorite free of taxation, except for his pledge to raise so

many troops on call. From this pledge the great lords have long since been released by processes of easy commutation. Even yet, they still hold half of England in their grip. Looked at from below, England is very poor. It is said that out of a hundred Englishmen only six make a last will and testament.[7] The rest have nothing to leave. One man in seventy holds about all that is worth having.

London is at once the poorest and the richest of all cities. Her East End is the hopper into which fall the incompetents of the land, the generations of those whom war could not use. At the same time she is the clearing house of the world. The traders of all nations meet there to balance their accounts. She is the center of the buying and selling of money,—but not the people's money.

The shadow of debt in England and on the continent grows with the growth of sea power and land power and imperial dominion. It looms darker still against a glowing background of pomp and circumstance. For the debt of the nation is the debt of the toiler. It is borne on the back of industry.

> "Fall to each, whate'er befall,
> The farmer, he must pay for all."

Behind and beneath all, public affairs stand the people. They do not count for much in great

7 Reginald J. Campbell at Ford Hall, Boston.

displays and their final end, according to Gambetta, is a "beggar crouching by a barrack door." Yet as soldiers and as taxpayers they are really necessary to the continued dominance of a great and fearless nation. An essential element in militarism is a patient industrial army which can pay the costs. It seems plain enough that the great lords cannot pay the taxes and that the great bankers will not. To be relatively tax-free is one of the natural privileges of greatness.

It is true that in England the lords are coming more and more to bear their share of the costs they help to create. In continental Europe tax discrepancies are greater. It must be granted however, that the world over, in America as well as in Europe, industry carries more than its share of the burdens.

"*Gare au bas vide*" (beware of the empty stocking), was a warning of Gambetta. To tax too closely is to risk the overthrow of organized government.

VI. "SYNDICATES FOR WAR"

It is a fact, more or less well known, that the arguments that "expansion of armaments is necessary to insure peace," that "big armies and navies are the insurance premiums of peace," and that "to insure peace a nation must always be prepared for war," rest heavily on the desires of the armament syndicates to keep up their business. The armament lobby of Europe is the most powerfully organized instrument of its kind in the world. Its operations are consciously and carefully planned. It is ably supported by a very large body of men and women consciously or unconsciously interested in one fashion or another in military expenditure. It has also the continual and effective backing of that class, in business or in journalism, who in Burke's famous phrase "scent with delight the cadaverous odor of lucre." The term "Armor-Plate Press" is effectively applied by Francis W. Hirst to the large group of subsidized journals. "*Defense not Defiance,*" says Robert Young, is the "international code-signal" of the armament pirates.

The British Ship Lobby

In a recent article,[1] Francis McCullagh of London describes the "greatest of the unseen and

[1] Syndicates for War; *New York Evening Post,* April 1, 1911.

pernicious forces with which economists have to contend." These are "the powerful companies which exist to produce armaments and which have been encouraged to increase their capital obligations within the last few years by the successive scares and naval programmes of the last decade." The capitalization of the six leading English firms is thus quoted from the *London Morning Leader:*

Vickers, Sons and Maxim	$ 40,000,000
Cammell, Laird & Co	20,500,000
Armstrong, Whitworth & Co.	33,500,000
Wm. Beardmore & Co.	18,500,000
John Brown & Co.	21,000,000
Thames Ironworks Company	4,300,000
	$137,800,000

This list is by no means complete so far as England is concerned. "The importance of these figures," says McCullagh, "is evident. The country has encouraged private concerns to expend these sums so that they may be productive of profits year by year for the benefit of their shareholders. Any restriction in the building of armaments either by the home or foreign Governments has disastrous results on the year's profits. It requires no stretch of the imagination to see that the enormous number of investors in every class of society scattered through the country exert a subtle influence in favor of the expansion

of armaments. The numbers are not so much as the quality." According to the *Investor's Review*, the social position of some of the leading owners of three of the principal firms is as follows:

	Vickers & Maxim	Brown & Co.	Armstrong & Whitworth
Duke	2	1	..
Marquis	2	..	..
Earl or Baron	50	10	60
Baronet	15	2	15
Knight	5	5	20
Member of Parliament.	3	2	8
J. P.	7	9	3
K. C.	..	..	5
Military or Naval Officer	21	2	20
Journalist	6	3	8

It is said that the plant of Vickers' Sons and Maxim is prepared to lay down and complete three dreadnaughts in three years without going outside its own factories.

Whatever the final effect on Great Britain or on civilization, these plants must be fed with government orders.

At the Lord Mayor's Banquet, in November, 1911, Mr. Winston Churchill, first lord of the Admiralty, is quoted as saying that "naval supremacy is the whole foundation of the British Empire. Upon it stands, not the empire alone, not merely

commercial prosperity, not merely a first place in the world's affairs, but actually our lives and the freedom we have guarded for nearly a thousand years."

If an independent and courageous leader in "Liberal" politics thus makes himself the mouthpiece for the armament syndicate, it is not surprising that minor officials should feel the same patriotic impulse. In the *London Nation* (March 9, 1912) we are told that two government officials "occupying distinguished and confidential positions" have joined the Armstrongs as directors. "Sir Charles Ottley was until the other day the naval Secretary of the Defence Committee, and in this and other high and confidential posts must have had access to all the inner secrets of our defensive services. Sir George Murray was . . . the Permanent Secretary of the Treasury and was therefore the official head of the Bureau which controls national expenditure. . . . The weight of experience, of confidential knowledge of social and official ties, which has thus been added to the resources of a firm competing for contracts could hardly be exaggerated. . . . Henceforth it will be in the minds of senior men that . . . such valuable appointments are open to them on their retirement. They must constantly meet, officially or socially, the agents of firms which might so reward them, and in their dealings the tempting thought can hardly fail to insinuate itself in their minds, that a public serv-

ant who stands well with a great contractor may look to him in his declining years for a very valuable and remunerative post."

Militarism Further Entrenched

In referring to the standing army of men now maintained by the British Empire, the "largest peace establishment" in the world, Mr. George Herbert Perris[2] says:

"And behind this force of able-bodied and middle-aged Englishmen, there lie two bodies, also of adult men, most skilled and able-bodied, whose numbers can be only approximately determined: (1) Those engaged in the arsenals and dockyards, and the numerous armament trades; and (2) Pensioners, small and large, possibly 100,000 of them, since their cost on the estimates is about £2,500,000 a year.

"The probability is, that at least 1,500,000 adult able-bodied men—or one in six of the 'occupied' adult males of the United Kingdom—share, to some extent, in the £65,000,000 a year which we spend on the twin 'defense' services. Thus, even when we remember that many of these, like the 'Terriers' and Reservists, get a mere allowance, while a large part of the regular army is paid for by India, it will be seen that we have here the most widely ramified of all our vested interests, a fearful drag upon reproductive industry, and an influence which must often

2 "Hands Across the Sea."

diverge from the straight line of democratic advance. The big prizes, of course, all go to a small class of financiers and industrial magnates, who, in order to keep the game going, exert a thoroughly pernicious influence on Parliament and middle-class opinion. The higher official ranks of the army and navy are an aristocratic preserve, and are highly organized for the advancement of their professional interests. This alliance of money power and class power, whose shibboleth and trademark is 'Imperialism,' includes the most determinedly reactionary element in British society.

"We are a part of a world-wide movement against obsolete forms of servitude, savagery and waste. The best of the civilization of to-day is on our side and the power of to-morrow is ours. Greedy contractors, silly scare-mongers, and their official friends whether in Germany or in England, are not checked by warlike preparations on the other side—quite the reverse. Each country must get rid of its own parasites. The democratic parties in each land must cut the claws of the enemies of the people. This is the work of national defence—the only road to real national security, the only true patriotism."

Activities of Armament Syndicates

It is stated on good authority that the establishment created by the late "King Krupp of

Essen,"[3] still the most noted of all builders of engines of war, maintains its ambassadors in every court of Europe. It is the business of these "strong, silent men" to force or coax the rulers of the nations into patriotic rivalry in the matter of buying great guns and great warships on credit. Behind them, still stronger and more silent men are prepared to loan the money needed on the terms of a moderate rate of interest and a cash bonus in advance. One of our greatest railroad builders used to boast that no one could trail him "by the nickels he had dropped." But it is claimed that the initiated can trace Krupp's men across the continent of Europe by their tips and *douceurs* as well as by the political downfall of the public officials not open to their persuasions. Not long since, according to McCullagh, the war minister of Servia was forced by Germany to resign because he had noted the personal interest of the German minister at Belgrade in the supplying of guns. M. Clemenceau intimates

[3] The directors of the Krupp Company have recently declared a dividend of 25 per cent.

The Austrian establishment for the manufacture of firearms ("Fabrique d'Armes Autrichiennes") has declared a dividend of 16 per cent. after a large addition to the reserve fund. The stock of all these companies is far above par, as far as that of the nations they rob is below. The larger the armament the more easily may further increase be secured. Each step widens the circle of bribery.

It is said that the imperialist journal *Ueberall* "carries sixteen pages of advertisements of the Krupp and Schican companies."

that, in Argentina, "French guns are beaten by the German because the emissaries of Krupp and his associates are more generous in their tips."

McCullagh also tells of meeting in Constantinople a military emissary selling arms to the Turks after putting through a good business in St. Petersburg. "At that moment," says he, "the young Turk officer was supposed to be so full of patriotism that he would cut your head off if you so much as hinted at bribery. But this astute military man from the North assured me that bribes were still accepted and still absolutely necessary. As a matter of fact, he bought up whole commissions of experts who were appointed to examine the weapons he had submitted."

"That all this diabolical activity," continues McCullagh, "makes for war is beyond all doubt. The good folks who sell Turkey a hundred million cartridges would not be averse to a Balkan scare or even to a Balkan war which would make Turkey want another hundred million to-morrow."

That in the United States similar activities are at work, both in and out of Congress, is a fact well attested although details are not easily secured.[4] Whenever the question of appropria-

[4] In different issues of the *New York Evening Post* of some four years ago is given a full account of the efforts at that time of one "General" of militia, editor of *Arms and the Man,* about the halls of Congress in the interest of various plans of military expenditure. Through his activity a "National Board for the Promotion of Rifle Practice"

tions is brought to the front we hear the same old stories as to the designs of Germany on the Monroe Doctrine, and the schemes of Japan on

was established. This Board maintained an active press agent, "to be paid for his work through voluntary contributions made by powder and ammunition makers and other persons and parties interested in the rifle practice propaganda." "The matter written by the Press Agent was distributed by the War Department and sent out free under the franking privileges."

The main purpose of the "General's" activities was to induce the Government to supply militia companies with guns, ammunition and trophies and especially to induce the purchase of ammunition from private companies.

The "General" greatly regrets "that the United States does not buy a considerable and fixed proportion of its ammunition from commercial manufacturers each year. Such a course would afford an invaluable means of comparing the merits of the respective creations and effect the stimulation of each by competition. . . . A similar method should be employed in obtaining our rifles." To this end he urges that besides the "School of Musketry at the Presidio of Monterey," we should have eight others, displacing "a system of target practice which is archaic."

The DuPont Powder Company issues a pamphlet on the "Policy of Patriotism," in which it is clearly shown that its factories "can be regarded in no other light than quasi-governmental institutions." "Approximately $300,000,000 is expended annually on the Army and Navy, and less than one per cent., or about $3,000,000, of this vast sum goes to powder. How important the item of excellence!"

A friend in Congress calls my attention to the following significant abstract from the Congressional Record, April 28, 1911, in reference to a Senator from Delaware:

"Mr. du Pont"

Assignments:

Military Affairs, Chairman.
Coast Defenses.
Expenditures in the War Department.
Pensions.

California or the Philippines. We are told of "35,000 Japanese ex-soldiers in Hawaii," of the purchase by Japan of Magdalena Bay "where she has already 75,000 soldiers," of the need of "meeting our enemies in the open ocean," of the "Danger Zone of the Caribbean Sea," and of other matters, real or imaginary, calculated to induce us to continue to "throw good money after bad" in the interests of naval preponderance.

The "War Scare" as a Weapon

The chief weapon of the Armament Syndicate, because the most effective one for persuading a nation to go more and more deeply into debt, is the "war scare." Always the one nation is pitted against the other. Always there is imminent danger from our neighbors. Awful revelations appear at critical moments. Not alone in Europe, where war scares have a mischievous diplomacy behind them, but also in the United States, the peace center of the world.[5] Curiously enough, the "war scare" appears also in Australia and New Zealand. No part of the world is more naturally immune from even the thought of war than New Zealand, but even here the emissaries of armament are active. Patriotic zeal calls for universal conscription and the "impending dan-

[5] "If our navy should shrink to lesser proportions and should be permitted to fall below the level of Germany, France and Japan, these nations would bully our commerce and insult our Monroe Doctrine whenever they felt like it."—(Republican Peace Committee, New York.)

ger of Japanese invasion" is urged with a vigor worthy of a real cause. Already Japanese fishermen have been seen on the reefs of New Caledonia, barely a thousand miles away! There is not the slightest evidence that Japan or anybody in Japan has any designs whatever on Australia or New Zealand. The whole agitation would be absurd if it were not thoroughly mischievous.

The storm center of war scares is found in England, not that her danger is greater, but because she has more builders of armament. It is in the interest of these men, not of the nation, that Great Britain shall have twice the "sea power" of any other nation. The growth of England's armament reacts on Germany, furnishing her diplomatists with an excuse for her extravagance in shipbuilding. A further excuse for her army excesses appears in the specter of Panslavism, which is also readily evoked.[6]

[6] In this connection, and making toward the same end, we have the joyous philosophy of militarism. Heinrich Leo (1853) prays cheerfully and unafraid: "May God deliver us from the inertia of European peoples and make us a present of a good war, fresh and joyous, which shall traverse Europe with fury, pass her peoples through the sieve and rid us of that scrofulous chaff which fills every place and makes it too narrow for others, so that we can again live a decent human life where a pestilent air now suffocates us."

Truer to fact is the following: "You have been made sick by tasting dangerous poison. Great soldiers have often told their men . . . that they have tasted the salt of life. The salt of life! . . . For it is nothing but the salt of death. It is a very subtle poison

Typical of the war scare is the following referring to the story "in the air," to the effect that Japan is buying Magdalena Bay as a coaling station.

"This we have to say, and we mean every word of it—that we desire no trifling or tampering by a foreign power with our neighbors on the Western Hemisphere that may prove a menace to them or to us, or that may in any way interfere with the declared principles of the Monroe Doctrine. We should not heed this hysterical peace talk that Asiatic missionaries and other well-meaning but deluded fanatics are giving us. We desire peace, but the air seems full of war. The public safety demands that our coasts and possessions be promptly and adequately fortified at strategic points, that the regular army be increased to its full strength, that our State militia be organized, drilled, and equipped, and that we should possess a most formidable navy to be prepared at a moment's notice for any and all contingencies." [7]

In Professor Grant Showerman's charming essay, "Peace and the Professor," this quotation appears, accredited to a modern "Cassius": [8]

"I am one of those who look for the simplest

which may lie hidden in the blood for many years. I believe it is a terrible thing!"—(B. L. Putnam Weale; *Indiscreet Letters from Peking.*)

[7] *Mexican Revolution and American Public Opinion*, p. 4, 1911.—William Temple.

[8] Nicholas Murray Butler.

motives of explanation of action or of conduct. My impression is that somebody makes something by reason of the huge expenditures in preparation for war. Have you ever noticed that about the time that the appropriations are under consideration in the House of Commons, in the Chamber of Deputies or in the Reichstag, or just before that time, hostilities are always on the point of breaking out in two or three parts of the world at once! . . . It might be worth while . . . to make some measurement of the sincerity and disinterestedness of the lively type of patriotism which accompanies these military and naval debates the world over. Is the propelling motive for them to be found in economics or in psychology? . . . While both these admirable sciences are represented in the make-up of that propelling motive, economics is not always the less important of the two."

Hon. David J. Foster, the late Chairman of the House Committee on Foreign Affairs, ventured to assert:

"I am absolutely convinced that there is a criminal conspiracy on foot for the purpose of bringing on a war between the United States and Japan. Thousands upon thousands of dollars are being spent to carry on this propaganda, and I am confident that the plans of these conspirators will unfold themselves before very long. I am convinced that this constant agitation for a war between the two nations is nothing but a

subterfuge employed by those people who are determined that this government shall build not less than two battleships each year. To endanger the friendly relations of two great nations in order that certain selfish interests may be gratified is nothing short of criminal."

"Looking back over 60 years," says George Heck [9] in London: "I can recall innumerable scares as to the sinister designs of some foreign country, scares which were as groundless as their recurrence seems to be inevitable. When imagination takes the form of fear, it becomes not a priceless gift, but a costly danger." [10]

[9] *The Nation,* London, February 10, 1912.

[10] Among the really sincere creators of war scares, and there are such, "General" Homer Lea, author of the "Valor of Ignorance," stands in a class by himself. Born in Denver in 1876, a Sophomore at Stanford University in 1900, a boy with dreams of Empire, he spent some time in Canton, becoming a member of some secret society of agitators. "Undersized and frail the little General in uniform of his own devising, overburdened by his spreading epaulettes, was always a figure of merriment to the scoffers, but the very qualities that dwarfed him in the eyes of his neighbors added to his stature when the uprising in China became an effective reality." (*San Francisco Chronicle,* May 2, 1912.)

His title of "Commander of the Second Army Division, holding the rank of Lieutenant General over these forces" (see *Who's Who,* 1911, p. 1129) was also "of his own devising," the "forces" so far as known being "broomstick companies of Chinese in empty squares and vacant lots" about Los Angeles. He was never connected in any way with the United States Army.

The "Valor of Ignorance" is a crude but clever echo of the military philosophy of Napoleon's times, with plans of

The following note appears in the telegraphic despatches of the week:

"The appearance of Germany as a possible supporter of Colombia is regarded here as the latest proof of the Kaiser's willingness to challenge the Monroe Doctrine. The history of Germany's intriguing to get a foothold on American soil, although it has not yet been written, is of course well known in naval circles here. The best-informed officers of the Navy have long been convinced that the steady increase in the German fleet has been aimed not at England and not at France, but at the United States; and that it is not Japan in the Pacific which we need watch most closely, but Germany in the Atlantic, and that it is with her rapidly increasing battle-ships that we shall eventually have to try conclusions." (*San Francisco Chronicle*, March 11, 1912.)

A few days later we read:—"Strategists in Washington have not the slightest doubt that the moment the spark in China is ignited, Japan will seize upon the opportunity to grab the Philippines, if indeed that is not one of her principal objects in endeavoring to precipitate an international war." (*San Francisco Examiner*, March 20, 1912.)

Still later, April 3, a climax is reached in the following from the *Los Angeles Examiner:*

"A confidential letter received by a diplomatic

imaginary campaigns to be executed by Japanese in California. It has no value from the military or political point of view.

official in Washington contains the startling information that the moment the United States intervenes in Mexico a Japanese force will attack the United States. The letter states:

" 'I know that Japan is ready to help Mexico. I know that there are over 60,000 Japanese in Mexico, and I know each one is well armed and is only waiting the word to join the Mexican army.

" 'I know that every Central and South American State will send its quota of men and money to help Mexico. The United States would win in the end, but it would lose its prestige.

" 'Germany would get a big slice of Brazil; France would get part of Chile, and England a part of Argentina. The Japs would get the Philippines and Sandwich Islands and would bombard the cities on the Pacific slope. I learned this from a source that is undeniable, but I determined that I would not tell it unless intervention was imminent.' "

On the same day, April 3, 1912, the following appeared in the *Boston American:*

"For more than three months a great syndicate has been in negotiation with the Mexican Government for a vast tract bordering on Magdalena Bay. The avowed purpose is to establish a Japanese colony. Back of it, however, is understood to be the Japanese Government.

"Already 75,000 Japanese are located along this most convenient body of water. Nearly every ship of the Japanese line, which operates on the Pacific coast, adds to the number. While many of these men are farmers, most of them are trained soldiers, many being veterans of the Russo-Japanese war.

"So great is the menace of this colony to the peace of the American continent, according to information received by the *American* to-day, that as far back as a year ago Great Britain sent a secret note to the United States demanding that the Oriental nation restrict its activities.

"In this note Great Britain called attention to the fact that she had an alliance for offence and defence with Japan and was therefore unable to make forceful representation herself. Such representation, the note insisted, must be made by the United States."

It may be noted that Magdalena Bay lies in the rainless belt of Lower California, that the lands about it are uninhabited and practically worthless, mountainous, and without vegetation save scanty cactus and cedar bushes. There is a good harbor, in a stormless sea. The fisheries in the roadstead are very rich, and are covered by a concession made by the Government of Mexico to Mr. A. Sandoval, resident in Los Angeles. This concession includes all the fisheries of Lower California.

There are a score or two of Japanese fishermen in Southern California and some of them have, I am told, examined the Sandoval concession. There is no market for fresh fish in Lower California, but there is a chance for salting large fish and for packing small ones in oil, as also turtle flesh and crabs at Magdalena Bay. There is no town[11] of any consequence nor

[11] As a matter of fact, there is now at Magdalena Bay

site for a town, as there is no water, save, I am told, from a brackish spring, opening among the sand dunes. On the Bay, there was once a colony who gathered from the rocks the lichen called Orchilla, then used as a yellow dye, but now replaced by the cheaper anilines. The whole orig-

a village of 100 people, six of them Japanese, a few Chinese, the rest mostly Mexicans. These are employed at a cannery owned by Mr. Sandoval. In this, crabs and green turtle are put up in tins. The flesh of the tunny is salted and dried while other fishes are made into fertilizer. It is expected that this concession will be ultimately developed with French, not Japanese, capital and with the aid of French fishermen. There are practically no Mexican fishermen, and since 1907 no Japanese laborers have been allowed by the Foreign Office at Tokyo to come to Mexico or to Canada.

A third enterprise of expatriated Japanese and their associates in Mexico has also come under the notice and condemnation of the "Armor-plate Press." A fishery concession covering some 200 miles of coast about Salina Cruz, another to the same extent about Manzanillo, and a third similar to these between these two points, near Acapulco, have been offered to bidders for a period of ten years, at a total rental of $1500 each. Representatives of the Toyo Hoge Kaisha (Oriental Whaling Company) have secured for a time the refusal of these concessions, at $2,000 for the three for a period of ten years. Along these shores fishes are abundant. They are not easily preserved by salting, especially with Mexican rock-salt. They dry up or else decay before the salt strikes in. The market for fresh fish is too far away, and the canning of sardines in that climate is a precarious business, as they are likely to spoil before they can be brought to land. The sums involved in the whole matter are petty, and these little ventures call for no notice from our Government. There is not a shadow of evidence that the Japanese Government or any combination of Japanese capital has ever had any designs on anything in Mexico.

inal basis of these newspaper stories is thus given by a Japanese friend, conversant with the facts, a man of the highest standing in San Francisco:

"There appears to be a concern known as the Chartered Company of Lower California, of which John E. Blackman of Los Angeles is President. I do not know where or how this company was incorporated, or where its headquarters are; but it seems to be in possession of a grant from the Mexican Government of certain lands on the west coast of Lower California, including territory on Magdalena Bay. Efforts to sell a portion of this land, or colonize it, seem to have been made by the company, and, among its other activities, it has attempted to interest Japanese capital. Through its representatives it approached Messrs. O. Noda and K. Abiko, two well-known Japanese residents of this city, and Mr. Noda was induced to go down last winter and inspect the territory. In doing so he represented no one but himself and possibly Mr. Abiko. Neither one of them has any capital behind him, nor has either one of them any authority to represent any bank, steamship or other financial body—let alone the Government. Acting in his personal capacity, Mr. Noda had a perfect right to take the course he did; but, in view of the readiness with which certain trouble-seekers in this country are prone to seize upon every opportunity to misrepresent and distort the motives of the Japanese Government and the individual enterprises of its people, his course was, perhaps, injudicious and unwise. It was an incident—perfectly proper in itself—but readily lending itself to the

purposes of mischief-makers and sensationalists. Mr. Noda is now conducting a little business in Sacramento and Mr. Abiko edits a newspaper in this city. So far as they are concerned the matter is ended."

Whether as a crab cannery or speculation in desert lands, the whole affair has no more international significance than would arise if "Italy" established a new peanut stand on the Bowery "in dangerous proximity to the treasures of Wall Street." [12]

The fear of Japan lends spice to journalism in other parts of the world as well. In the *Japan Chronicle* (March 21, 1912), I find a translation of an article in the Russian journal *Dalny Vostok*. The writer, who signs his name as "Dalinsky," sees in the awakening giant of China and the mighty military power of Japan a menace to Russia making it "necessary to take steps beforehand against a new Mongolian invasion."

He finds that "at the present time Japan for warlike purposes can dispose of the following forces:"

[12] Yet, at Washington and in the "Armor-Plate Press" we are told that the Magdalena Bay "Colony" is a menace not only to the United States, but to the Panama Canal. It is located at the same distance from Panama that it is from Pittsburgh. There is no fuel nor food (except fish) obtainable in quantity from any point nearer than Mazatlan or San Diego, both as far away as Boston is from Washington, with no regular connections.

With the colors	763,000	men
First reserve	415,000	
Second reserve	831,000	
Soldiers	2,009,000	men
Trained militia	124,000	
Untrained militia	873,000	
Militia	997,000	
Total	3,006,000	

These, "with the already existing Korean troops," give to the cry of "Asia for the Asiatics!" a most powerful backing, and Russia may indeed tremble. "We have been keeping ourselves quieter than water, lower than the grass . . . giving way to all their wishes, even trying to anticipate them. China is still weak, and a prey to civil war. America, until the digging of the Panama Canal, is helpless against Japan." "A crow does not peck out a crow's eyes." . . . "In this outburst (of aggressive movement) Japan will probably come into collision with us and not with the Chinese."

It is evident that the imagination of the "dockyard strategist" is quite as vivid and fantastic in Russia as in Washington. How pitiful seem the 75,000 armed Japanese gathered by Mr. Hearst on the barren sand dunes of Magdalena Bay as compared with the three millions thus conjured up by this Russian operator!

The *Sydney Bulletin* in Australia, as quoted in the *Japan Chronicle*, cites the case of one Yang Ki-tak, a Korean imprisoned for some reason by the Japanese, and described as a member of "a race which believed in the talk of Peace Societies and spurned the idea of naval or other defense."

It says:

"Australia's personal interest in the above lies in the fact that the fate of Yang Ki-tak of Korea may be the fate of John Brown of Victoria or Bill Smith of New South Wales, if the Great Trouble comes before the Commonwealth Boy Army has had time to grow up or its infant fleet a chance to develop into something worth while."

Henri Golay (Berne, 1912) says: "The actual political philosophy of government is fear of ghosts. Each state has two or three ghosts which hold the strings to which are attached the ministers and chiefs of that state. When one lives under the fear of ghosts, one exists in a distorted and unreal light—the veritable molehill would appear as a mountain, the little ship quietly steaming off on a voyage would suddenly appear to assume the proportions of a pirate, and one word uttered from a mouth more or less responsible is enough to call forth a declaration of war. Let us as nations cast aside our ghosts . . . and disarmament will come as a natural sequence."

Sir Edward Grey once observed that many misunderstandings could be averted by "an interna-

tional exchange of journalists." An exchange of "dockyard strategists" might also help.

Armament for War or Peace?

Certain general propositions may be laid down as to war and peace. To regard war as "glorious" is to invite it. To regard it as a hideous calamity is to avoid it. The nation which holds war in the background as a possibility in case of difference is likely sooner or later to resort to it. The nation that eliminates war from its methods of adjustment will find peaceful methods at hand whenever differences do arise. To be pledged to arbitration is to be "thrice armed," for it is to have one's "quarrel just."

Artificial "ententes," meaningless friendships and "entangling alliances" increase the danger of war. To a nation's own enemies and the friends of its enemies, these coalitions add the enemies of its friends. The diplomacy which seeks to thwart the aspirations, righteous or otherwise, of other nations is itself thwarted in turn, and each entanglement is described in terms of war.

It is the claim of each nation that its armament is solely for defense, for the protection of its commerce, its colonies or its coasts. This claim is clearly not true in every case; perhaps not true in any case.

Purely defensive armament may make for peace. It may impress rival diplomatists that war would be a risky process. But defensive

armament is never satisfied to remain purely defensive. It is intolerant of a waiting policy. It tends to aggrandize itself. It would be "prepared to meet the enemy in the middle of the sea." In other words, it would become armament for offense as well as defense. War involves both.

Moreover, the fighters are dissatisfied with a waiting game. "We like to have a mark that will wriggle when we hit it. We cannot have weapons without wanting to try them and see whether or not they will work."

Professor William Graham Sumner [13] has well said:

"There is no state of readiness for war; the notion calls for never-ending sacrifices. It is a fallacy. It is evident that to pursue such a notion with any idea of realizing it would absorb all the resources and activity of the state; this the great European states are now proving by experiment. A wiser rule would be to make up your mind soberly what you want, peace or war, and then to get ready for what you want; for what we prepare for is what we shall get."

"War," says the German Colonel Gadke, "is the father of other wars. The more we think of our own power and ability, the oftener we have tasted of the fruit of victorious war, the more we are surrounded by the evil spirit of Chauvinism and of imperialism."

The "Peace Establishment"

The military equipment of England is often

[13] *Yale Review*, October, 1911.

mentioned as "The Peace Establishment." Here we have a stroke of giant humor. Thoreau once referred to the weather as being "so dry, it could fairly be called wet."

But it is nevertheless true that the so-called "Peace Establishment" does make for peace in two ways. It may promote the "Peace of Preponderance" to use Lord Rosebery's phrase, or what we may call the Peace of Impotence.

So far as "preponderance" goes, it is clear that a little nation will not often attack a big one whatever its grievance. Persia will not declare war on Russia and England, nor, if you please, Colombia on the United States. It is fairly safe to be "preponderant" but that fact has nothing to do with justice. [14]

The "Peace of Impotence" follows an overdrawn bank account. This introduces a third element, the international banker. His relations to war and peace have been already sufficiently discussed in these pages.

"Peace Establishments" and Secret Diplomacy

In the Europe of to-day, "Peace Establishments" serve mainly as counters in the game of secret diplomacy. When the chief business of the state was war, the function of the diplomatist was to spy out weak places for attack in his

[14] "Where empires towered that were not just
Lo the skulking wild fox scratches in a little heap of dust!"

—(Lowell.)

neighbors' preserves, next to devise pretexts to make such attack seem honorable. The armament builders of the world strive to keep alive this tradition. The presence of idle armament is a constant spur to diplomatic activity. Thus is established a mischievous round of diplomacy and armament in which every "Great Power" is entrapped. "It has been apparent," says Admiral Mahan, "that the governments of Great Britain, France and Germany have earnestly striven for peace." On the face of things, such would not appear to be the fact. Rather the center of war menace seems to lie in the Foreign Offices of these three nations. If private citizens behaved toward their fellows in like fashion, it would be said that they were "hunting for trouble."

Great armaments support a policy of diplomatic aggravation. Diplomatic aggravation in turn serves to make costly armament inevitable.

Economic Difficulties in Disarmament

Under present conditions with an armament expenditure for the world of over $2,000,000,000 yearly, the number of men engaged runs far into the millions. With the suppression of war all of these men as well as all the other millions now withdrawn by conscription, will be returned to the ranks of toilers.[15]

Such a transfer will temporarily involve seri-

[15] After the Boer war, according to McCullagh, the extra hands employed in the Government gun-factories at

ous disturbance in economic adjustment. For it is recognized that any fundamental change in labor conditions causes immediate confusion to some part of the industrial world. The advent of peace would for a time bring widespread distress, although it would soon be more than compensated for, as the mass of the people would thus be released from the burden of supporting millions in what (economically considered) is destructive idleness. When these also became workers the aggregate of demand and supply would rise alike, and all would gain from the shutting off of the avenues of waste.

During the period of transition, however, there would certainly be suffering. Would it be unreasonable to expect the commonwealth which has forced its artisans into gun-making, powder-making and the building of warships, to lead in amelioration of conditions following the change of military establishments into industrial and the makers of war implements into artisans of peace? The change must come—will come—and it is for wise statesmanship to make it as little drastic as possible. As Governments have reserved to them-

Woolwich had to be discharged. "What else was to be done? The Government could not proceed to start another war just for the sake of keeping these men in employment, and it could not pay them for being idle. Nevertheless a roar of indignation went up from the imperialistic press." Even the labor leaders, or some of them, joined in this clamor against throwing so many good men out of employment merely because the Government had nothing for them to do.

selves the right of making war, they should acknowledge the obligations incident to its passing. The revival of the American Merchant fleet would go far to relieve the exigencies connected with the abandonment of war preparation in the United States. As to our naval personnel, the changes impending must of necessity be so gradual as to cause no serious embarrassment. Our standing army is small, being relatively near a peace basis. Its extravagant cost is due not to its size, but to the lack of discretion on the part of Congress in the needless multiplication of army posts.[16]

16 See page 168.

VII. WAR TO-DAY

Types of Modern War

In our time, the various wars actual or threatened may be divided into three types. These we may term Civil War; International War; Imperial War.

Civil war is strife within the confines of a nation. It indicates, in general, the failure of the essential attribute of a modern nation,—the will and the power to maintain peace within itself. Internal peace depends on the maintenance of a representative legislative assembly on the one hand and of courts of justice on the other. A despotism is essentially a condition of civil war.[1] Historically, in civil war every despotism has found its final and necessary end. Civil war may be the work of thieves, marauders and malcontents; or it may be the last resort of "murdered, mangled liberty"; or it may also be an inextricable mixture of the two.

International war is a conflict between recognized nations, more or less equal in prestige. This form of strife has filled the pages of European history, and it stands as a constant menace in the background of most of the diplomacy of the world.

[1] A Chinese saying thus describes absolutism in China: "Big fish eat little fish; little fish eat shrimp; shrimp eat mud."

Imperial war, or in Novicow's classification, "War of Spoliation," comprises conflicts undertaken for extension of national territory and rule. Its function is the control by a strong hand of inefficient or inadequate peoples. Its motives may be various, half-altruistic or wholly selfish, for riches or display. The results may be equally various, moral and physical improvement of the conquered nation, or the utter ruin of its people through processes of miltary pacification, or any intermediate combination of the two. Whatever the motives, it is worth notice that the real one is seldom actually avowed. Always these conflicts take the guise of benevolence or else of "military necessity" or "manifest destiny." It is said that the Turks are the only people who do not try to veil in some way their schemes of territorial spoliation.[2]

The only check to civil war consists in the establishment of democracy. Constitutional government and the development of the courts afford its only remedy.

International war, as I have already tried to

[2] The following sentences from speeches of Abraham Lincoln are pertinent to the discussion of Imperial Control.

"Let us discard all this quibbling about this man and the other man, this race and that race and the other race being inferior and therefore must be placed in an inferior position."

"What I do say is that no man is good enough to govern another without that other's consent."

"The sheep and the wolf do not agree on the meaning of liberty."

make clear, is now virtually at an end. The growth of debt and the cost of armament have placed the control of European affairs in the hands of the money-lenders. It has forged for them weapons against which the belligerent nations are powerless. The more warlike the nation, the more firmly is it held in the actual grip of debt.

With Imperial war the case is not so clear. It is likely to continue until all barbarous lands are covered by the flags of civilization. And the fierce intrigues and bitter jealousies to which these operations give rise hold within them again the menace of international war. But here too the bankers and the common-sense of the people may intervene. Imperial wars are scarcely less costly than others, and almost as profitless. Meanwhile the public opinion of the world is rising against and increasingly condemns the encroachments of Europe on Asia and even on Africa.

"The Great Illusion"

Norman Angell has applied the term, "The Great Illusion" to the idea that a nation can be strengthened or enriched by war. War of any sort no longer pays. Conquered territory is always a burden of expense. It adds to national vigor or to national wealth only when it becomes an integral part of the nation, the home of a coöperating and self-governing people; the "Louisiana Purchase" of the United States rep-

resents an expansion of this kind. Illustrations of costly but unprofitable holdings may be seen almost anywhere on the map of Asia or Africa. A well-governed dependency may be a source of moral strength to the parent nation but not a source of wealth. There can no longer be "taxation without representation." A "colony" cannot safely be asked to pay even its military expenditures. The secret of our creditable recent government of the Philippines lies largely in the fact that the taxes of the islands are used for their own public purposes. For the services of army and navy we do not ask them to pay. Our own military outlay in their behalf has been estimated (1912) at $168,000,000. Nowhere in America or Europe have the people had the experience of being taxed for civil expenditures only. In England, as Mr. L. T. Hobhouse observes:[3] "The proceeds of expanding revenue flow into the War Office and the Admiralty, and it is with difficulty that social reformers snatch something on the way. . . . It is this constant increase which hamstrings measures for the internal development of the country, and cripples every effort to alleviate the widespread misery of the masses."

For behind the "Great Illusion" looms the "Great Debt," and both are necessary results of the intricate and futile diplomacy which underlies the affairs of Europe to-day.

[3] *Atlantic Monthly*, March, 1912, p. 351.

The "Mirage of the Map"

By the "Mirage of the Map"[4] is meant the fallacy that national importance is measured by extent of territory. "In the days" says Norman Angell, "of the sailing ship and the lumbering wagon dragging slowly over all but impassable roads, for one country to derive profit from another it had to administer it politically. But the steam engine, the railway, the telegraph have profoundly modified the problem. In the modern world political dominion is playing a more and effaced rôle in commerce. It is the case with every modern nation that the outside territories it exploits most successfully are precisely those of which it does not own a foot. Even with Great Britain, the greater part of her over-seas trade is done with nations which she makes no attempt to "own," control, coerce, or dominate. She has ceased to do any of these things with her colonies. . . . The modern German exploits South America by remaining at home. German colonies are colonies '*pour rire.*' The Government has to bribe Germans to go to them. Her trade with them is microscopic. If the 20,000,000 people who have been added to Germany's population since the war had to depend on their country's political conquest, they would have had to starve. . . . Germany draws more tribute from South America

[4] Norman Angell, *London Daily Mail,* November 14, 1911.

than Spain which has poured out mountains of treasure and oceans of blood in its conquest. These (South American states) are Germany's real colonies. . . . The immense trade they represent owes nothing to the diplomat, to the Agadir incidents, to Dreadnaughts. It is the unaided work of the merchant and manufacturer."

In carrying the French flag and the French debt over Algiers, Tunis, Morocco, Sahara, and Soudan,—about two-fifths of all Africa,—France has gained nothing in population or wealth. After thirty years in Tunis, she has established a colony of 25,000 Frenchmen. This exactly corresponds to the yearly loss of population in France, "the real France," which grows less every year. "The diplomats can point to 25,000 Frenchmen living artificially and exotically under conditions inimical to their race, as expansion and as evidence that France is maintaining her position as a great power. . . . There are to-day more Germans in France than there are Frenchmen in all the colonies that France has acquired in the last half century and German trade with France outweighs enormously the trade of France with all French colonies. France is to-day a better colony for Germans than any exotic colony which France owns.

" 'They tell me,' said a French deputy 'that the Germans are at Agadir. I know they are in the Champs Elysées.' " And Paris is a much better place for doing business than Morocco.

Professor Delbrück, a leading German Imperialist, in the "Preussische Jahrbücher," explains and defends the spirit of the "mirage of the map." Germany we are told desires not isolated over-seas possessions but "a vast continuous area which is purely German." It must stretch somewhere from ocean to ocean and it must be ruled by an "aristocracy of German planters and merchants" which will evolve "a German-African national pride." It is freely admitted that no economic gain can arise from the control of this domain. The desire for it rests solely on racial feeling. Mr. H. W. Massingham from whom we have taken the above quotations compares this to the involuntary reflex movements of an animal organism deprived of its directive nerve centers. "Nations continue to act on a given impulse long after the living thought of a people has advanced beyond the motives which inspired the action, precisely as a frog will go on moving after its brain is removed. . . . To be a governing aristocracy among naked tribes and to see German descend to be the patois of semi-civilized negroes is not a dream which allures us by its æsthetic charm. Already it admits its own intellectual nullity. It has shed its economic fallacies. It has dropped the argument from material advantage. It stands frankly on a basis of sheer sentimentality." (*The Nation*, London, March 16, 1912.)

The Hidden Trail of Diplomacy

Through all these matters of colonial expan-

sion, there runs a sinuous trail of diplomatic intrigue. The secret treaty[5] is the bane of Europe, because the people have no control over it, and its existence is made known only through its unpleasant reactions. In physics it is recognized that a current of electricity will set up counter currents in conducting bodies which lie parallel to it. Something of this kind takes place with the currents of intrigue. Parallel with the main line of diplomacy and contrary to it are two other lines, each of the same nature as the first.

The main trail leads from Berlin to Vienna, then down the Adriatic and through the Balkan capitals to Constantinople. Here effort centers in the Great Bagdad Railway with its direct threat at India, to which it is much the shortest route, and its indirect menace to Egypt on the one hand and to Persia on the other. This great military highway built by a German Syndicate on a concession from the Turkish Government to the German Empire was to lead across the Taurus Mountains down the rich but wasted valley of the Euphrates, past Bagdad, Nineveh and Babylon to the Persian Gulf.

To the eastward runs another hidden line of intrigue from St. Petersburg across the Caucasus,

[5] One of the wisest provisions of the Constitution of the United States is that requiring confirmation of treaties by the open vote of the Senate. This involves delay, an element of safety, and it insures publicity, which makes diplomatic intrigue an impossibility.

involving Persia, Turkestan, Mongolia, North Manchuria, and whatever else is exploitable along the whole Russian frontier. To the westward, a similar line of secret agreements extends from London to Paris, Rome and Athens and across to Delhi, coöperating at times with the underground schemes of Russia and at all times crossing and thwarting the secret designs of Germany. Two lines of intrigue met at Morocco, and in the arrangements of both it seems to be agreed that Italy for her part should at her convenience take possession of Tripoli. In the one stroke of the annexation of the half-forgotten principality of Koweit,[6] the diplomacy which we may call British

[6] The essential facts in the Koweit affair are as follows: The Sultan Abdul-Hamid had granted to the government of Germany a concession of land on which to build a railway from Constantinople diagonally across Turkey to the Persian Gulf. This would cross the Taurus Mountains to Adana and Aleppo, then proceed to Nineveh on the Tigris, thence to Bagdad and Babylon to Koweit. The concession also included navigation rights on the Tigris, Euphrates and Shat-el-Arab, besides access to the richest of oil lands and of wheat-fields in the Valley of Mesopotamia. Koweit, on the Persian Gulf, is an "insignificant cluster of mud huts." Its Sheik, a half-independent Arab vassal of the Sultan, was visited some years ago by a British officer. According to report the gunboat which brought him from India was generously provided with presents, as well as with flagstaffs and the Union Jack. It did not take long to persuade the Sheik that it was for his advantage to come under British protection. In this fashion was Koweit annexed to the British Empire. Thereupon the German Foreign Office was promptly notified that the terminus of the Bagdad Railroad would be on British soil. It was therefore suggested that the road

achieved a distinct advantage, unless we count the sacrifice of moral values involved in assisting at the loot of Persia. Koweit constitutes the only possible terminus for the Bagdad Route and when it passed under the British flag the costly railway was halted [7] at the Taurus Mountains. By further consenting to the effacement of Persia, England has saved that nation from the danger of German domination by turning her over bodily to the military despotism of Russia. For it is claimed that a self-governing Persia "would be simply a plum to be picked by the owner of the Bagdad Railway at his convenience." A secondary ramification of diplomacy is the fair-weather alliance with Japan. Another is the zeal for conscription aroused in Australia and New Zealand.

be made neutral, British and Germans to share in its management and construction. Work was immediately stopped and the line still extends no farther than Eregli, in the Taurus Mountains. If completed it would form a route to India much shorter than that by the Suez Canal. It was therefore regarded as a "military menace" to the British occupation of India, as well as to Egypt and Persia. It is not easy for a layman to see any reality in this alleged "menace." Besides the reasons for believing that a war between Great Britain and Germany is impossible, we have the difficulties involved in a very long range attack on an entrenched antagonist. The German army would be needed at home. If ever an attack were to have been made on England, it would seem as if the time of the Boer war offered the best opportunity. It was probably morally, as well as financially, impossible then, as we trust that it is to-day.

[7] In essentially the same fashion, the British "Cape-to-Cairo" line has been blocked by German influence in Africa.

In the war-scares evolved in these regions, Japan, four thousand miles away, is the only evocable spectre.

"Powers" or Jurisdictions?

The mediæval conception of a nation, still extant in Europe, is that of a "Power," its importance being determined by the physical force it can exert on other nations.

Our ultimate conception must be that of a simple jurisdiction either as an individual unit or else as composed of confederated states. In this view, the boundary line represents merely the limit of jurisdiction. That jurisdiction ceases does not imply probability of violence between the people on the two sides nor require fortification for the purpose of repelling violence. The Canadian boundary is an example of this meeting of states not as powers but as jurisdictions.

This four-thousand-mile line, ranging through all kinds of territory and all sorts of conditions, disputed nearly all the way "with all the brutal frankness common to blood relations,"[8] has for nearly a hundred years not known a fortress, a soldier, a warship or a gun. It is a peace boundary, the limit of the jurisdiction of one self-governing nation, the beginning of that of another. It lacks but one thing to make it ideally perfect, the removal of the custom-house, the emblem of national suspicion and greed, the remnant of the

[8] Dr. James A. MacDonald.

days when it was considered good economics for a nation to "have its taxes paid by foreigners." Entirely similar to this in all regards is the long boundary of Mexico. This we have to defend, it is true, not against Mexicans but against the predatory excursions of our own law-breakers.

The states in a federal union meet as jurisdictions. The small ones have no fear of the large ones and those not touching the sea suffer in no way from their restricted position. A "Power" hampered as is the state of Illinois would chafe against its limitations, and its jingoes would talk of fighting their way to the ocean. But viewed as a jurisdiction, surrounded by similar jurisdictions, the people of Illinois have no consciousness of their limitation.

Germany a Case in Point

Viewed as a "Great Power," in the mediæval fashion, the German Empire is hampered on every side. Her scant sea-coast is split in two by the presence of Denmark. Her German Rhine discharges itself through Holland. The ports of Amsterdam, Rotterdam, Antwerp and Ostend, geographically hers, are occupied by alien people whom she could crush out in a moment, were it not for the physical force of the rest of Europe and the moral force of the world. Of Poland, she has too much or too little.[9] A large part of

[9] "Wir haben Polen genug auf dem Wagen." (A German Journal.)

the German people live in the alien empire of Austro-Hungary and in the republic of Switzerland, while after forty years of possession, she scarcely owns Alsace or Lorraine. She is hemmed in everywhere by the scars of old struggles, to her perennial discomfort. For this reason, she suffers from the "Drang nach Osten," she seeks a road to the Persian Gulf, an empire over seas, and every form of imperial extension to lands "under the sun," which may for the moment seem plausible or possible.

But Germany as a *jurisdiction* suffers none of these limitations. Her powers are only those which are needed for the people's good. They are merely her public duties. It matters nothing that her sway is checked on almost every side before it reaches the sea. Other jurisdictions intervene, and each of these looks after the public needs of man, which are mostly justice, conservation, education, sanitation and peace. As one of the "Great Powers" of the world, Germany (with her fellow states as well) is a center of friction, injustice and unrest. Viewed as a jurisdiction, the administration of Germany is worthy of the highest praise, and no enlargement of boundaries would enhance her usefulness. When the nations cease to be "Powers," great and small, and become fellow-citizens of a civilized world, this fact will mark the reign of peace.

The power and extent of a nation bear no relation to its prosperity, that is, to the welfare of

its individual citizens, except as the debt and waste through which "power" establishes itself may be harmful to the people of which the nation is composed. "There is no welfare of a nation apart from the well-being of its people." [10]

England and Germany as Rival "Powers"

A conspicuous example of the rivalry engendered between great nations still viewed as "Powers" is found in the present relations of England and Germany. Between these nations as intelligent, civilized and progressive peoples there is no reason for enmity. As a matter of fact, except as an aftermath of war scares and serpentine diplomacy no such enmity exists. No responsible person on either side, King, Kaiser, Reichstag or Parliament, desires war. All realize that a fight between these two most powerful, most determined and most advanced of nations would involve the entire world in indescribable calamity. And we know that the whole influence of commerce, of labor, of education on both sides is unalterably opposed to armed conflict. Behind all this, as we believe, the Unseen Empire of Finance, the natural order of business, has decided that such a conflict shall never take place.

Nevertheless, the old idea of the function and duties of a "Power," with its burden of debt, its necessity for armament and its mischief-making diplomacy, has kept these two nations for years on the apparent verge of war. Some

[10] Eikichi Kamada, President of Keio University, Tokyo.

phase of this menace will endure until we clarify the common conception of what a nation is for.

Why Talk of War?

The trouble in each nation comes from the alleged wickedness of its neighbors. Each tells the same story, "The waning fleet of England is tied to its shores by German menace"; "The small but efficient fleet of Germany is solely for defense of Germany's growing commerce." More explicitly, on March 16, 1911, there was prepared the following statement signed by Eickhoff, chairman of the German Committee for "Peace and Arbitration," also endorsed by fifty-one members of the German Reichstag and thirty-four members of the Prussian Diet, as well as by the Count of Schwerin-Lowitz, President of the Reichstag and by the Prince of Schoenaich-Caroleth.

The statement (translated) reads as follows:

"It is the opinion of the undersigned that because of the interests involved in the close international interdependence existing in the finance and commerce of the world, Germany holds its army and navy in readiness only as a means of protection against any attack, but not as a means for aggressive warfare (nicht aber um einen Angriffskrieg zu führen)."

In England it would be easy to get an array of distinguished names after any statement of similar tenor. It could be done in any nation. War is the last thing the statesmen want, although

it is the first thing for which they prepare. Neither nation could gain much by victory, while in the end there would be no material choice between that and defeat. The British would still own England, the Germans Germany, whatever the trend of events. Not an Englishman (contractors, armament-builders and "ghouls" excepted) would be the richer for the downfall of Germany. Not a German but would be the poorer for the destruction of England's credit.[11]

The toilers of the world are everywhere opposed to war and to debt. The Labor Unions are almost a unit in this regard. The Socialist groups

[11] "How far are we removed from the glorious days when Bismarck could glibly talk of bleeding France white with the satisfactory assurance that not a German would be the poorer in consequence, and that on the contrary the German state would immensely gain thereby." By "the social Law of Acceleration . . . Bismarck was nearer to being able to apply the methods of Attila, nearly 1,500 years removed from him, than we are to being able to apply the methods of Bismarck, from whom only 40 years separate us. . . .

"But surely we must realize that in the end the Government *is* the world of affairs, in the sense that the general trend of its policy must sooner or later be determined by the interests and the necessities of the mass of the people from which it derives its power, its money, its general capacity to act with efficiency and precision—a modern war of all things involves that capacity which must be derived from acting in the long run in connection with the great currents, economic and moral, of its time and people. It is not possible for any great state taking an active part in the life of the world to do otherwise. The state simply is powerless before these currents."—(Norman Angell, "Influences of Banking on International Relations.")

take the same stand against international war, though many of them advocate another kind, the struggle of "class consciousness." The Social Democracy of Germany,[12] pledged in opposition to monarchy as well as to militarism, has arisen as an inevitable reaction from the policy of "the Mailed Fist." As for the rank and file, they are mere incidents in the movement of diplomacy; not the principals, merely the victims or, at most, pawns to be moved in the game of national glory. Hidden forces control their activities, and their lives are spent to no purpose known to themselves. The Prime Ministers and Ministers of War are never on the firing line. The pawnbrokers, promoters, and camp-followers are also well to the rear. To risk their own lives would also risk the profits. "The rich man's war, the poor man's fight, this," says Carl Schurz, "was the Winged Word" among the poor whites of the Tennessee mountains.

Professional Interest a Factor

There can be no doubt that in every navy, many of the wardroom officers look beyond their dreary routine of futile repetitions to the time when they can make use of their costly

[12] In the recent German election, January 12, 1912, the Social Democrats polled over seven millions of votes as against less than five millions cast by other parties. In every district but one in Berlin including Potsdam itself, they were successful. The result is significant of the great reaction in Germany.

training. So in looking forward to the day of action there may be first a selfish element. War means a chance for promotion, a chance to rise from the nameless roll into the rank of the idols of the hour.

A higher impulse arises from the professional spirit. When a man devotes his life and energy to a calling, he may naturally wish to practise it, in a degree at least. It is true some of the ablest advocates of peace are found among men who know what war is and who realize that it must disappear in the onward movement of civilization. But the average European officer feels the spirit of militarism. He takes his profession seriously, and at a value set by the money expended on it. For the cost either in money or in blood, he cares little. Of the cost in money, he knows little: he has nothing to do with taxes. Of the cost in blood, he knows little more, only a very small percentage of European military men having ever seen a battle. Militarism has always held itself superior to civil affairs and lightly regarded the life of the individual. Napoleon is reported to have said: "A soldier like me cares not a tinker's damn for the lives of a million men."

There exists moreover the scientific desire to test, in some way, the teaching of the books in military affairs. Changes in armament mean great changes in tactics, but these have never been formulated or tested. They can be tested only in the laboratory of actual war. It is there-

fore not strange, and it is perhaps no reproach to the young officers that they toast, "Der Tag," as the day of their release and their opportunity.

A Grotesque of History

The persistence of mediæval traditions of war and glory in an age of science, commerce and reason has produced an extraordinary confusion of effort and motive without a parallel in the history of the world. This condition can be viewed only as a part of the necessary stages in the passing of war.

Zangwill says, quoting from a supposititious future-day Chinese historian: "Like a sloughing snake, the West lay sickening; the new skin of commercialism only half put forth, the old skin of militarism only half put off. A truly piebald monster, this boasted civilization of ours. On the one hand, a federation of peoples eagerly strengthening one another; on the other hand, packs of peoples jealously snapping at one another. A sextet of nations styling themselves Great Powers, all with vast capitals invested in developing one another's resources, were yet feverishly occupied in watching and cramping the faintest extension of one another's dominions. A more ironic situation had never been presented in human history, not even when Christianity was at its apogee. For, whereas, in the contest between Church and Camp, it was simple enough to shelve the Sermon on the Mount; in the contest

between Commerce and Camp both factors were of equal vitality and insistence. The result of this shock of opposite forces of development were paradoxical, farcical even. In the ancient world there had been the same struggle for supremacy, but the Babylonians or the Egyptians did not build up each other's greatness. The Romans did not lend money to the Carthaginians, nor did Hannibal sell the Roman elephants. But in this era the nations fought by taking up one another's war loans. In lulls of peace they built for one another the ships they would presently be bombarding one another with. The ancient mistress of the world never developed a country till it belonged to Rome. The mediæval rival mistresses were all engaged in developing countries which belonged to their rivals, or to which they might one day themselves belong. In brief, two threads of social evolution had got tangled up and tied into a knot, so that neither thread could be followed clearly. It was death to give away your country's fortifications—at a percentage. It was high treason to help the enemy in war time, but you could sell him your deadliest inventions if your Government offered less or waived you aside. And you manufactured those weapons and exported them to the enemy by the million so long as he had not given you notice that he was going to fight you next week. Quite often a nation was hoist with its own petards and no sooner had you devastated your enemy's country

than you lent him money to build it all up again. In vain shells hissed and dynamite exploded. The stockbroker followed ever on the heels of the soldier and the grass of new life (and new loans) sprang up over the blackened ruins. Indeed, nations, instead of being extinguished in the struggle for political existence, because they were too weak to pay their debts, had to be kept artificially alive in order to pay them."

VIII. RETRENCHMENT

In these pages we have tried to indicate the present relations of the nations to war and peace. We have shown how peace has followed in the wake of debt, and how the rule of the war-lord has given place to that of the money-lender, and this in turn to the operation of the ordinary laws of finance. We have shown that nations have lived beyond their means to a degree that cannot be permanent. We shall further see that the demands of peace must arise to contest with war the right to "the fruits of progress." The way into debt lay through heedless extravagance. The way out lies through a reversal of policy, a return to forethought and frugality.

Functions of Government

Until well along in the nineteenth century, war and diplomacy constituted practically the sole business of government. The force of tradition still gives them the right of way in almost every country. The present organization of every government in the world is based on the mediæval theory [1] that war is a natural function of a nation, not a moral, physical and financial catas-

[1] The conception was thus formulated by Machiavelli: "A prince ought to have no other aim or thought, nor select anything else for his study than war and its rules of discipline, for that is the sole art that belongs to him that rules."

trophe. In our own country, the President is Commander-in-chief of Army and Navy, with a Secretary of each in his Cabinet. It is not humanly possible for any one of these three to reduce the range of power his office represents. In almost all other nations the force of militarism is still more dominant. For these reasons, competing civil interests have never been able adequately to establish themselves anywhere. The sums devoted, the world over, to all civil purposes range in amount from one-fifth to less than one-half those devoted to military affairs.

A remedy for all this lies in the exaltation of the civil functions of government, the safeguarding of the common rights and interests of men in the promotion of justice, sanitation and education and in the conservation of public properties, together with the fostering in all effective ways of art, science and invention. Positive efforts along these lines rather than direct attacks on the waste of militarism may indicate the way out. The formation of a high commission which in statesmanlike fashion can consider actual facts in the scheme of national defense, with the relative values of those needs on which money may be expended, will give the surest solution of our problem in America.

Expenses Unchecked

One of the most disheartening features in popular government everywhere is the absence of all

machinery to check expense. Even an empty treasury does not count. There is always some way to negotiate a loan. This absence of check is especially notable in the case of the army and navy. Their expert demands are beyond the understanding of the common man. By ministries and parliaments their estimates are accepted practically without question. No finance minister anywhere has thus far maintained himself against them, with the single exception of Yamamoto in Japan.

The "watchdog of the treasury" receives no support from his fellows or his constituents, and his work for retrenchment marks him for political extinction.[2] Thus it happens that the United

[2] This is illustrated in the recent experience of Wermuth in Germany: "But the guardian of the finances of the Empire, Herr Wermuth, has gone, without a word of thanks for his services from his Imperi master. . . . The civilian goes and the warrior remains. There is something symbolical in this for the political situation of Germany and of the world of so-called Great Powers. . . . The true reason of Herr Wermuth's resignation lies deeper. It is not the dispute about one particular tax. Enough has leaked out to make it evident that the whole system of spending money and preparing further expenditure as practised at present has found a decided opponent in the former Chancellor of the Exchequer. He was certainly not in principle an antagonist of armaments; but he appears to have fought strongly against the reckless increase of expenditure on them. . . . The story of the naval madness in Germany furnishes an instructive example of how certain plants themselves create the conditions of their growth. With the increase of the German navy, the number of officers of the navy has increased too, so that to-day perhaps five or eight times more middle-class families have

States in time of peace, with nothing save peace ahead, spends $800,000 a day in military operations,—not counting interest and the removal of men from productive enterprise. No clear evidence of the need of this has ever come before the people. The world over, as we have shown, each nation finds itself in the same ruinous competition, with the same ignorance of any reason why these conditions should exist.

It is a reproach to the people of China that, suffering as they have for ages from the misuse of irresponsible power and from the form of taxation known as the "squeeze," they have until lately never attacked the system, rising only at intervals against the individuals who have abused it. The same reproach holds against the people of Europe. They have floundered for a century in the morass of debt, war-waste and poverty, but while there have been bread-riots and tax riots, strikes, "sabotage" and "syndicalism," the people have made no adequate stand against the

sons or near relatives in the navy than fifteen years ago, and consequently take a much larger and more direct interest in all questions referring to it. This accounts also for the small resistance offered by the middle classes to the proposed increase. Hence the only quarrel of the middle class parties in regard to this question relates to the method of paying for it." (Ed. Bernstein, London *Nation*, March, 1912.) With each expansion of naval armament goes a corresponding extension of the numbers of those consciously or unconsciously interested in promoting naval expenditure, and this influence is cast on the side of still greater debt and waste.

system of military expenditure which is one of the primal sources of economic troubles.

The United States and Retrenchment

The recent "Pageant of the Ships" on the Hudson River was regarded by some of us as a mighty illustration of the force of a free people, a great awakener of patriotism, and a glory of democracy. To others it represented the most costly monument ever reared to the spirit of heedless waste. There is perhaps some middle ground on which we may meet.

No one may question the necessity for a small but effective army and a navy also small but with the best of necessary equipment. There is certainly some justification for protecting our great coast cities, but many of us see no reason for fortifying the Panama Canal nor the port of Honolulu. There is no nation on earth that has either the will or the money to attack us. On the other hand we see in international friendship, international commerce and freedom from international debt, weapons stronger than any fortress or navy. If in America we can safely retrench, if we ought to retrench and dare to retrench, it will ultimately solve for the world the whole problem of debt and militarism.

Let us consider this matter somewhat in detail. Civil wars aside, our nation has been three times engaged in war, each time of its own initiative. Leaving out the War of 1812, in some

degree an aftermath of the Revolution, we have only the wars with Mexico and with Spain. The Mexican war no one now pretends to justify. It was an affair of spoliation in the interests of slavery. That beneficent results followed in no way justifies its purpose.

The primary motive in our differences with Spain was one of sympathy for a much abused people. The usual processes of "military pacification"[3] familiar in Asia and Africa and going on to-day in Persia, Armenia and other unfortunate regions, were being practiced at our own doors. It was natural and righteous to protest. But this need not have involved war. It certainly did not involve war, for it is a matter of record that before war was declared, our Minister to Spain had secured assent to a treaty granting all we had a right to ask, all in fact that we did ask,—namely full autonomy to Cuba and arbitration of all differences, including the disaster of the Maine.

[3] For a statement of facts as to "military pacification" in some other regions see "Bella! Bella! Horrida Bella," a masterpiece of moral indignation, by Mr. F. J. Corbet, in the *Westminster Review,* March, 1902. Also "Pax Britannica in South Africa," by Francis P. Fletcher-Vane, and "The Captain Sleeps," in the present author's "Imperial Democracy." For the close relation between a standing army and the "White Slave Traffic," see "Queen's Daughters in India," by Elizabeth Andrews and Katherine Bushnell, and also Mr. Corbet's article mentioned above. See also "Indiscreet Letters from Peking," by B. L. Putnam Weale. "History," says Mr. Weale, "is made, only to be immediately forgotten."

Whatever else we may say of that war, we cannot assert that before it began, we were exposed to the danger of Spanish aggression.

We are not exposed to aggression from any quarter. As the wealthiest nation on earth, we have business allies in every capital. In spite of our tariffs, we are the world's best customer. Unlike the great nations of Europe, we have not overdrawn our accounts. We spend as much each year as the best of them—or the worst—but our debt is small and our reserve surplus is gigantic. If we had no navy whatever, no nation could afford to attack us. For we are protected in isolation, in wealth, in relative freedom from debt, in the absence of entangling alliances and entangling enmities, in having no fallen dynasties to restore and no defeats to avenge, in being—as a whole—fully competent, through commerce, through industry, through alliance by blood and alliance of friendship to banish all consideration of war.

Moreover, the United States has a defense in her moral strength, an influence strained a few times to be sure by lapses real or apparent, yet on the whole persistent and potent. To have stood fairly consistently for peace and fair play is in itself a guarantee against wanton attack even if we had no other defense. Were the United States to make war on any power of Europe (a wholly absurd proposition), it could not under present conditions invade the enemy's territory,

nor inflict much damage on its people or its property. The United States in wealth, population and resources stands far above the strongest of the European powers, but she could not successfully attack any of them on their own ground. Conditions of warfare have changed since the days of Napoleon, and Napoleon fought on land because by his own avowal he could not control the sea. The most the United States could do was exemplified in the war with Spain. She could, pirate-fashion, capture a few merchant ships, doing some injury to her enemy's commerce and to her own (carried mostly in European bottoms) and she might dislodge the occupants of a remote dependency. All of which is based on the imaginary possibility that this nation may again forget herself and resort to the crude and brutal arbitrament of war, when so many other ways are simpler, cheaper and more honorable.

Furthermore, under any conditions, the United States, with or without a navy, is in its turn effectively defended against any attack from Europe.

In a recent article,[4] Captain Alexander G. McClellan, an English authority, expresses strongly his belief that naval expenditure is a menace to the nation responsible for it and that the United States, for example, would be amply defended against any attack from Europe, even with no navy at all.

[4] A British View of American Naval Expenditure, *Atlantic Monthly*, January, 1911.

The whole coast-line is admirably adapted for defense and our important harbors are protected by shore guns which no fleet could face or silence. Even should these fail, no fleet would dare enter a harbor guarded by submarine vessels, torpedoes and floating mines, with purposely misplaced buoys and beacons. No city of importance stands on an unprotected bay, and no enemy's squadron would waste its precious strength on fishing villages and seaside resorts. No nation could maintain a successful blockade and none could do much damage in the robbery of our merchant ships, because most of these, thanks to our protective shipping laws, already fly the British or the German flag. For any nation to land and maintain an army on our shores, as Captain McClellan clearly shows, is quite impossible. "The nation which could fight a war like the Civil War, without even a standing army worth speaking of and divided against itself, has little to fear from any army of invasion, even though it should gain admittance into the country."

Captain McClellan further observes: "My arguments are logical, and therefore I ask: Is America justified in spending about $150,000,000 yearly on her navy, when the most powerful antagonist that we can put against her cannot do damage enough to require that sum to set it right again, in one year? I think not! . . .

"But the question whether the American fleet could be destroyed or not, could not in the least

affect the final result, when one takes into consideration the infinitesimal amount of damage which an enemy's fleet could do, were there no American fleet on the spot to stop it. That small damage in no way justifies America's present naval expenditure, or even the existence of her navy at all.

"Nothing in the world can justify America in building a fleet strong enough to tackle Japan single-handed on the Pacific, or a fleet strong enough to tackle single-handed any European naval power on the Atlantic. It would mean keeping her navy up to a two- or three-power standard all the time. Will American extravagance run to this? If not, why play at owning a navy to satisfy vanity? Why pay away one hundred and fifty millions of dollars a year on the navy when it is practically helpless, because it lacks the vital support of a merchant marine? The all-around-the-world trip which an American fleet made a couple of years ago would have been an impossibility without the help afforded by British and German colliers. Not one American merchant-Jack's ensign could be seen in attendance on the naval ships during the whole cruise. This was commented upon by the chief in command—Admiral Evans. Not very palatable reading, is it? Remember, it applies to the country with the finest navigable coasts, harbors, and rivers in the world!

"There is still another important point of view

to consider, and it is this: Britain's and Germany's merchant marines are chiefly composed of deep-water—foreign-going—ships, while American merchant ships are chiefly engaged in the coastal and inter-coastal trade. Again, Britain depends upon her merchant ships for the means to live; America does not. In case of war and blockade, her coasters could tie up in harbor, coil down their ropes, and wait for peace. The work they do could be carried on by railroads.

"Turn now to American commerce. Here lies another great advantage of America. She can afford to stand by and snap her fingers at any nation, no matter what the size of its navy. In the first place, her position as a producer makes her absolutely independent of all nations: other nations must come to her, and not she to them, for necessities. This being the case, she is in a position to retaliate without firing a shot, should offensive measures be taken against her. Again, where two such countries as Britain and Germany depend upon America for the employment of a great part of their shipping, war with either is a remote possibility. America, not owning a deep-water merchant marine, need fear no captures or destruction in this direction. Should America carry on a war with Germany, what would happen to her over-sea commerce? Simply nothing! During these times of too much merchant tonnage, British ships would be only too glad to take American products anywhere; and so would

German vessels in case of an Anglo-American war. Thus we see that if America went to war with either country, the damage would be confined to a few unimportant towns on the coast, and her over-sea commerce would reach its destination just as merrily as ever. Peace also has its victories, and the country which warred with America would find that after war had ceased, her ships would have little left to pick up in the way of cargo. A revival of old trade relations would not come with the declaration of peace, but it would take years of keen competition to regain the lost ground.

"Would it not be better if America voted less on naval ships and just a little on merchant ships? The latter would bring millions into the treasury, while the former only take millions out. It would prove a profitable investment, I am sure.

"If in the march of civilization we need the help of battle-ships and 12-inch guns, then I say that our civilization is rotten, and will not last. I am confident that the day is not far off when the people of America, at least, will oppose the needless waste of millions. The preparations for war which need never come about, only suggest childish folly which must be thrown aside.

" 'Mailed fists' and huge standing armies and navies are out of date. . . . As a plain sailor who has seen all the mighty navies of the world, I say in plain language that they stand only to mock us and prove our civilization a sham.

. . . The man in America, or even in Europe, who thinks that this craze can last, or is bound to culminate in a war, has a poorer opinion of his fellow men than I have."

From these and other considerations, it is evident that no foreign nation could inflict on the United States an injury comparable to the cost of the present upkeep of the American Navy. It is also evident that a nation attacking the United States would work, through derangement of finance and commerce, an injury to itself greater than it could inflict on us.

We have next to inquire from what direction we may expect attack. We are on the best of terms officially with every nation. No nation for half a century has ever seriously suggested making war upon us. We have had in the past no foreign wars save of our own choosing. There was never a time when our people were so determinedly set on peace. Where then is our antagonist?

Not Great Britain, that is certain. We are bone of her bone and Canada, between us, is drawn by the closest ties to both. There have been in the past English statesmen, scornful of plebeian America, and American politicians whose function it was to "twist the British Lion's tail." We have been sometimes swayed or apparently swayed by Irish influences. We have been vexed by "a certain condescension from foreigners" and we have resented the bombast of British Imperial-

ism, too much like our own. But all this has been skin-deep, and the ties that bind us to England may not be severed. With or without a treaty, we shall never have a difference with her that cannot be settled by arbitration.

Is it France? Assuredly not. Here again we have a multitude of ties and no hint of estrangement. No armament-lobbyist or other disturber of society has ventured to suggest a possible clash between the two great republics.

What then of Germany? Germany and the United States have a thousand reasons for friendship and not one for enmity. Nearly one-fourth of our people are of German blood and Germany has furnished full share of our leading men. Our higher education owes more to German scholarship than to all other foreign influences whatsoever. The interweaving of business and commerce between Germany and the United States is scarcely less complex than the nexus which joins us to England. The great commerce represented by the North German Lloyd and the Hamburg-American Company deals chiefly with the United States. England's quarrels are not ours. Moreover, we do not believe that England has any real quarrel with Germany.[5]

[5] In the midst of futile noise and nagging diplomacy, it is most refreshing to read the noble and scholarly address delivered by Viscount Haldane, British Secretary of State for War, at the University of Oxford on August 3, 1911, entitled "Great Britain and Germany, a study in National Characteristics." This is a most truthful and sympathetic

But the Monroe Doctrine, some may say, does not this gall Germany by restricting her colonial operations to Africa and Asia? If so, Germany has given no indication of being vexed. Commercially she controls South America already, if we may use the word "control" for a relation which rests on mutual benefit. Her merchants have seen their opportunity and they have used it. The same chance is open to any nation. The German trade would gain nothing from the annexation of any South American state.

But South Brazil is becoming German. Will not these provinces revolt and throw themselves under the protection of the German flag? There are no visible signs of any such intention, no likelihood of the dismemberment of Brazil, no desire to become part of German Imperialism. The "mailed fist," and the "spiked helmet," the emblems of German imperialism, have no attraction to the German trader. He prefers to carry on his business under a less exacting flag than that of the Fatherland. I have yet to find a German who attaches the slightest importance to this matter. It is certainly too remote to enter into our military and naval calculations. Even if it demanded consideration, the boycott is a weapon more effective than the sword. But we may note in passing that sword and boycott alike are two-edged, cutting both ways at once.

discussion of the real relations, altogether friendly, of the people of the two nations.

Is our enemy among the smaller nations of Europe? To ask the question is to answer it. The one item of the tourist trade goes far to pardon any lapses on our part in international courtesy.

Is it Russia? We deprecate the Russian treatment of Jews, of Persians, of Finns—and for that matter of Russians—but no one dreams of fighting about it. At long range, moral suasion is our strongest weapon.

Have we an enemy in South America? No military expert will ever admit it, but every commercial expert recognizes that we might to advantage go farther in the promotion of friendships with these Latin nations.

Only Japan is left and to point her out as our rival and our enemy is the most criminal folly of all. The Japanese are fond of saying: "The Pacific Ocean unites our nations, it does not separate." The leaders of Japan are largely men educated in the United States, and thoroughly imbued with our traditions of college loyalty. Japan recognizes the United States as her nearest neighbor among Western nations, her best customer and her most steadfast friend. It is said of Korea that through the ages her face was toward China, her back toward Japan. Through the ages, the face of Japan has been toward Asia. Her present ambitions and her financial interests lie now in the reclaiming of Korea, in the safeguarding of her investments

in South Manchuria and in the part she must yet play in the future of China. For her own affairs she needs every *yen* that she can raise for the next half century. She would not if she could organize an expedition against us and she could not if she would.

A Way to Retrench

We believe that the public opinion of the nation is turning away from war. This means sooner or later an abatement. But in our present machinery of government there is no point where public opinion impinges on the matter of national expense. That this is a "billion-dollar nation," spending that incalculable sum each year on its civil and military affairs, seems a matter rather of pride than shame, even though we are told on very high authority that one-third of this sum is absolutely wasted.

Following the model of our Tariff Board, there should be in the United States a High Commission composed of statesmen and economists who should decide, as civilian citizens, on the aim, extent and purpose of national defense. This done, a committee of military experts should determine ways and means to accomplish these necessary results.[6]

[6] Since writing the above, I have learned that the Secretary of War has suggested a somewhat similar plan. This involves the creation of a National Council of Defense, composed of two members of the Cabinet, four Senators, four Representatives, two Army and two Navy officers. It shall be the duty of this Commission to sug-

Besides the gigantic cost of armament there is another sort of waste in connection with national defense, for which the officials of army and navy are not in the remotest degree responsible. This consists in extravagant multiplication of navy yards and army posts at the demand of local pride or local greed, through the efforts of "log-rolling" Congressmen.

General W. W. Wotherspoon[7] estimates that the elimination of those useless expenditures which have been forced on Congress by local demands would effect a reduction of more than half the present cost of maintenance of the army. A proper condensation of army posts would reduce the cost of these establishments from $42,300,000 to $21,132,000 and the annual cost of upkeep from $846,720 to $211,680. A similar reorganization of the Navy Yards, as urged by the Secretary of the Navy, would effect a still larger reduction.

An officer of the army once said to the writer: "It is no part of the work of military experts to determine what places need defense, but to spend

gest "a broad and comprehensive military and naval policy for the country." In the composition of this body, it is said "there lies no desire that military opinion should prevail in its deliberation."

The appointment of such a Commission would be a great step toward making our military expenditures rational as well as economical, though it would seem to me that the non-official public should also be represented, and perhaps by a majority in the commission.

[7] *The Independent*, February 15, 1912, p. 343.

to the best advantage whatever sums may be assigned to any given purpose. If we are directed to fortify Honolulu, we shall plan the most perfect defense possible for the money and under the conditions."

Readjustment of Values

It is necessary that we should undertake a readjustment of our estimate of relative values in our functions of government. Matters of the most vital moment pass unheeded, while sums which would meet almost every need which government can satisfy are voted for items of militarism, apparently without a second thought.

For example, we have in the band of "white slavers" [8] an international group of the worst kind of criminals who form a menace to every family in our land. The sum now provided yearly for protection against them would keep us in smokeless powder for a day and a half. The whole infamous traffic the world over could be exterminated for the cost of a third-rate battleship.[9]

The enemies that threaten our society are within, not without. The artistic completeness of the work of defending our shores against impos-

[8] The term "White Slave" was originally and aptly applied to military conscripts, according to Frédéric Passy. It was used in 1867, by Emile Girardin, and it originated with the Emperor Napoleon III.

[9] At the present writing, this sum is said to be less than $15,000. However, in the current appropriation bill of 1912, $250,000 has been asked for.

sible enemies should, if necessary, wait until we can make some better start on our moral and physical sanitation. The danger from foreign foes is a mere nightmare reminiscence of mediævalism. The danger from the "White Slaver" and the "Red Plague" surrounds us on every side.

In this connection we may mention the need of a national university. Every great capital in the world, London and Washington alone excepted, maintains as a matter of course, a correspondingly great university. Since the days of Washington who gave his fortune to Congress to this end, the project has been under discussion. With each succeeding Congress it has been set aside because the millions necessary could not be spared from the cost of national defense.

Such a center of advanced work is in itself a national defense. It would utilize for our enlightenment the great riches of our capital in science, art, statistics and libraries. It would furnish a reservoir of experts who could be drawn upon in all activities of government in which special training is required: it would call from other lands advanced students of the workings of democracy. Lastly it would furnish to the social life of Washington the element not now adequately developed, of men of character and scholarship permanently at home, to whom the petty incidents of politics and the fate of appropriation bills are not the objects of first interest.

In the recasting of values the need of a national university should have a leading place.

It should be the place of the university at Washington to demonstrate with Professor Huxley that the day has come when civilized nations must "discard their old weapons to make way for the new ones forced upon us by the growth of knowledge and the rush of commerce." "We are in the presence," Professor Lockyer [10] reminds us, "of a new struggle for existence, a struggle which once commenced must go on till the fittest survive. It is a struggle in which science and brains take the place of swords and sinews. The school, the university, the laboratory and the workshop are the battlefields of this new warfare."

The real strength of a great nation does not lie in its belated militarism, developed at the cost of an unceasing burden of debt. It rests on the lives and character of its men, their ability to work to the best advantage morally and economically and their power to assimilate and to utilize the garnered and classified experience of the race.

In his famous address on the "Moral Grandeur of Nations" at Boston in 1845, Charles Sumner showed that the battle-ship *Ohio*, then at anchor in Boston Harbor, had far exceeded in cost the greatest of our educational institutions. The en-

[10] *Nature*, Sept. 1903.

dowment of Harvard College at that time was $703,175, its annual budget $47,935. The *Ohio* had cost $834,845; its annual upkeep was $220,000.

Harvard University, still our greatest seat of learning, has grown immeasurably in six decades. Her endowment has now risen to $24,323,618, mostly from the grateful gifts of those who have been her students; her annual expenses to more than two million and a quarter. She has in fact far outrun the *Ohio* of 1845, though I believe that a new *Ohio* of costlier build is still in the service. I do not know what the new *Ohio* cost, but it will take half the endowment of Harvard University in a few years to replace it, while the entire endowment of Harvard College in Sumner's time would not pay its upkeep for a single year. And before long it must go unused and unremembered to the junkheap, while the influence of the University has permeated and permanently ennobled Christendom. This influence in all its ramifications in America to-day has a value beyond all question or comparison.

The need of battle-ships may be great—as to this we have yet to be convinced. But there can be no question as to the need of universities.

Chief Business of Government

The chief business of government should be no longer war and diplomacy. It should be, to fall back on the definition of Aristotle, restated in

similar terms wholly independently by Abraham Lincoln, to establish justice among men and to do those things of common necessity which collective action can accomplish better than private enterprise.

But in the past and in the present, it is certainly true that the most costly and most absorbing business of government has been war preparation. Government by Congress or by Parliament is still too often a device by which the people pay for what they do not want, and at times for what they do not get. "I cannot help thinking of you as ye deserve, O ye Governments," said Thoreau. "The only government that I recognize, and it matters not how few are at the head of it or how small its army, is that which establishes justice in the land, never that which establishes injustice."

The chief "national defense" which any nation needs to-day, is protection from the enemies within itself. Such protection is possible only through a broad statesmanship which sees the end from the beginning, and from beginning to end will strive for the welfare of its people.

The "grandeur of nations" is measured not by their extent on the map; not by their population or wealth nor their apparent military or naval supremacy; nor yet in the long run by their universities, their arts or their sciences. Emerson says "America means opportunity." That nation is great which to its rank and file means

opportunity and which further breeds men capable of seizing the opportunity it offers. First then they must be free from crushing burdens of debt. Next they must live in peace. War implies everywhere a reversal of the processes of natural selection. Broadly speaking, in war the strongest are destroyed, the men best fitted to be the parents of the new generation.

IX. THE PASSING OF WAR

The Future of War

In the majestic work of Jean de Bloch [1] on war and its future, the foundation of the modern peace movement, it is shown that international war has become a physical impossibility. The term war in that sense, as Bloch himself observed,[2] does not apply to "frontier brawls nor to punitive operations or trumpery expeditions against semi-barbarous people."

"The war of the future, the war which has become impossible, is the war which has haunted the imagination of mankind for the last thirty years, the war in which great nations, armed to the teeth were to fling themselves with all their resources into a struggle for life and death. This is the war that every day becomes more and more impossible . . . alike from a military, economic, and political point of view. The very development that has taken place in the mechanism of war has rendered war an impracticable operation. The dimensions of modern armaments and the organization of society have rendered its prosecution an economic impossibility, with finally the inevitable result of a catastrophe which would destroy all existing political organi-

[1] In Russian, Ivan Stanislavich Blioch.

[2] In an interview with Mr. William T. Stead.

zations. Thus, the great war cannot be made, and any attempt to make it would result in suicide."

Bloch concludes his discussion with the demonstration that:

"If the present conditions continue, there can be but two alternatives, either ruin from the continuance of the armed peace, or a veritable catastrophe from war.

"The question is naturally asked: What will be given to the people after war as compensation for their immense losses? The conquered certainly will be too exhausted to pay any money indemnity, and compensation must be taken by the retention of frontier territories which will be so impoverished by war that their acquisition will be a loss rather than a gain.

"With such conditions can we hope for good sense among millions of men when but a handful of their former officers remain? Will the armies of Western Europe, where the Socialist propaganda has already spread among the masses, allow themselves to be disarmed, and if not, must we not expect even greater disasters than those which marked the short-lived triumph of the Paris Commune? The longer the present position of affairs continues the greater is the probability of such convulsions after the close of a great war. It cannot be denied that conscription, by taking from productive occupations a greater number of men than the former conditions of

service, has increased the popularity of subversive principles among the masses. Formerly only Socialists were known; now Anarchism has arisen. Not long ago the advocates of revolution were a handful; now they have their representatives in all parliaments, and every new election increases their number in Germany, in France, in Austria and in Italy. It is a strange coincidence that only in England and in the United States, where conscription is unknown, are representative assemblies free from these elements of disintegration. Thus side by side with the growth of military burdens rise waves of popular discontent, threatening a social revolution.

"Such are the consequences of the so-called armed peace of Europe—slow destruction resulting from expenditure on preparations for war, or swift destruction in the event of war—in both events, convulsions in the social order."

A Way Out

The way out of war will open, the world over, with the enlightenment of public opinion, with the extension of international law and the perfection of the International Courts at The Hague. This machinery of Conciliation is created by public opinion, and with its more perfect adjustment, the force of public opinion behind it will grow steadily more insistent. Little by little in the thought of men war is erased from the list of possibilities. Its crude and costly conclusions

become less and less acceptable, and the victories of peace more and more welcome, and more and more stable.

The fact that a better way of composing differences exists is, in itself, a guarantee that no serious differences shall occur. For, as a rule, wars do not arise from the alleged "causes of war." These "causes" are almost wholly mere pretexts after war has been determined on. "Affairs of honor" between nations are worthy of no more respect than "affairs of honor" among men. In either case, an adequate remedy is found in a few days or months of patience, and in the adjustments of disinterested friends. This we call arbitration, and its supreme virtue with nations as with individuals, lies in its being unlimited.

In our own country, at present, there opens a door of escape from the waste of war preparation. This, as we have already suggested,[3] lies in the appointment of a civil commission which shall give a definite purpose to our plans of national defense. No one can justify gigantic expenditures, blindly undertaken. It is surely not necessary for us to strive for ideal perfection of defense against unknown and imaginary foes. It is surely unnecessary to pour out $800,000 a day (not counting pensions nor interest) simply because two other nations are doing the same, and still three others would keep step if

[3] See page 167.

they could. Nor should we act from year to year on the advice of interested parties solely, "muddling along" through sheer inertia without a look forward to our final aim.

Such an aim a commission of statesmen could furnish. With its help we should justify our ways or else change them. No one can doubt that to justify we must needs also change, in what way or in what degree perhaps no one can now foretell. But this at least is certain: if the United States should find for herself a definite policy, building no more fortresses, dreadnaughts, or destroyers until her best minds are convinced that these are needed, such action would go far, very far, toward solving the problems of debt-ridden Europe.

The Passing of War

The passing of war is marked by many conditions both incongruous and disconcerting, as I have already tried to set forth. From the standpoint of Social Evolution, these erratic and fantastic phenomena are all necessary stages in a world process—the change from the rule of force to that of law. On the one hand we note the persistence of mediæval traditions and their consequences, the burden of debt, the unwieldy and ruinous body of armament, the "war scare," the overlordship of the "pawnbroker," the sinuous trail of secret diplomacy, the

"Great Illusion" and the "Mirage of the Map."

On the other hand, and parallel with these, we remark the fraternity of trade, the unification of banking, the internationalism of art, science and invention, the steady extension of humane sentiments and the crystallization of world congresses and world courts. It has been observed that the different nations of Europe have yielded up their sovereignty and that they are now but "Provinces of the Unseen Empire." This phrase referred to the subservience of debt, but it is true in another and more honorable sense. They are in fact but provinces in the unseen empire of civilization. The world has become an intellectual unit. The thoughts of all men are the common property of all. In like fashion the world has become an economic unit. The currents of business flow through all nations alike. Whatever disturbs one part of the organism affects all others. The boundaries of nations really signify no more than the boundaries of counties or states. Only our outworn diplomacy and the enmities it engenders serve to conceal this fact.

It is easy to see that these are days of transition. The past is losing its hold. The future has yet to make its grasp complete. And from the larger point of view we see that these various conditions could not have come together at any earlier stage in the history of the world. A hun-

dred years ago, these combinations would have been unthinkable.

A hundred years hence, the combinations of to-day will be equally incredible. The motives behind our present war preparation will then seem as remote as to us now are the motives behind the great Crusades.

Mankind does not linger over impossibilities. The coat-of-mail vanished from European history all at once, when men realized that it had no further effectiveness. The war equipment of to-day will disappear scarcely less promptly when men see clearly the changes which have made it futile and absurd. In the fine and true words of Admiral Winslow: "No matter is so trivial that nations will not go to war over it, if they want to go to war. No difference is so weighty that it cannot be quietly settled if nations do not wish war."

Science has slain War. Rather it has forged the weapons by which War has slain itself. It remains for Finance to give it a decent burial.

APPENDIX

TABLES OF DEBT AND EXPENDITURE

Following are tables[1] illustrating the cost of armament and other matters of expense of the leading nations of the world. Of these, tables A to M are the work of Mr. Arthur W. Allen, certain additions to table A having been made by Mr. Clayton D. Carus.

[1] These tables have been separately printed by the World Peace Foundation under the title of "The Drain of Armament."

TABLE A.

THE INTEREST BEARING DEBTS OF THE PRINCIPAL NATIONS.

Country	Date	National debt	Approx. annual interest charge	Figures obtained from:
Austria-Hungary	Jan. 1, 1911	$ 3,612,389,000	$ 144,496,000	Almanach de Gotha '12
Belgium	do	740,681,000	21,249,000	do
Bulgaria	do	122,040,000	5,992,000	do
Denmark	April 1, 1911	90,682,000	2,545,000	do
France	Jan. 1, 1910	6,286,435,000	192,762,000	do
Germany	do	1,224,158,000	41,981,000	do
German States	do 1911	3,607,061,165	132,942,000	do
Great Britain	April 1, 1911	3,389,577,000	101,060,000	do
Greece	Jan. 1, 1911	155,823,000	6,233,000	do
Italy	July 1, 1909	2,614,183,000	92,145,000	Statesman's Yearbook, 1911
Netherlands	Jan. 1, 1912	465,295,000	12,886,000	Almanach de Gotha, 1912
Norway	July 1, 1910	86,386,000	3,024,000	do
Portugal	Jan. 1, 1911	818,578,000	28,650,000	do
Roumania	April 1, 1910	315,966,000	12,639,000	do
Russia	Jan. 1, 1911	4,507,071,000	180,283,000	do
Servia	do	135,886,000	6,115,000	do
Spain	do	1,886,221,000	75,448,000	do
Sweden	do	145,105,000	5,079,000	do
Switzerland	do	24,360,000	853,000	do
Turkey	Sept. 13, 1911	508,981,000	20,359,000	Rept. of Sir Adam Black 1911
Totals		$30,736,878,165	$1,086,741,000	

United States	July 1, 1911	915,353,000	21,311,000	U. S. Treas. Rept. '11
Canada	do	336,268,546	10,290,696	
Argentina	Jan. 1, 1911	531,858,000	26,593,000	Almanach de Gotha '12
Brazil	do	654,303,000	32,715,000	do
Chile	do	175,000,000	8,750,000	do
Colombia	do	16,622,000	831,000	do
Costa Rica		19,963,924	295,113	do
Cuba		48,296,585	1,980,507	
Ecuador	July 1, 1910	22,000,000	1,100,000	do
Guatemala		19,095,801	1,700,865	
Honduras		121,666,775	10,486,678	
Mexico	July 1, 1911	219,537,000	10,977,000	do
Paraguay		5,027,141	(payment ceased)	
Peru	1909	8,400,000	462,000	
Salvador		12,035,397	1,812,665	
Uruguay	Jan. 1, 1911	134,229,000	6,711,000	do
Venezuela	Jan. 1, 1911	39,300,000	1,179,000	
Australia		1,184,192,157	29,594,251	
China		601,916,605	92,375,016	
Japan	April 1, 1911	1,325,198,000	59,312,000	Almanach de Gotha '12
Korea		18,297,238	1,280,806	
New Zealand		346,439,001	10,645,075	
Persia		26,534,500	1,326,475	
Siam		19,466,000	875,970	
Grand Total		$37,537,877,835	$1,419,346,117	

TABLE B.

THE WORLD'S MILITARY EXPENDITURE.

England and the Continent of Europe.

Country	Fiscal year	Expended for army	Expended for navy	Total military charge (exclusive of interest, pensions, etc.)
Austria-Hungary	1911	$ 73,513,000	$ 13,731,000	$ 87,244,000
Belgium	1911	11,987,000	....	11,987,000
Bulgaria	1911	7,928,000	....	7,928,000
Denmark	1911-12	6,053,000	3,044,000	9,097,000
France	1911	[1] 187,632,000	83,286,000	270,918,000
Germany	1911-12	203,938,000	114,508,000	318,446,000
Great Britain	1910-11	138,800,000	203,020,000	341,820,000
Greece	1911	4,262,000	1,703,000	5,965,000
Italy	1911-12	[2] 81,033,000	39,643,000	120,676,000
Montenegro	1911	38,000	....	38,000
Netherlands	1912	12,120,000	8,146,000	20,266,000
Norway	1910-11	3,798,000	1,460,000	5,258,000
Portugal	1910-11	8,592,000	3,997,000	12,589,000
Roumania	1911-12	13,856,000	....	13,856,000
Russia	1911	265,642,000	54,128,000	319,770,000
Servia	1911	5,402,000	....	5,402,000
Spain	1911	37,671,000	13,696,000	51,367,000
Sweden	1912	15,314,000	7,251,000	22,565,000
Switzerland	1911	8,785,000	....	8,785,000
Turkey	1911-12	42,071,000	6,223,000	48,294,000
Totals		$1,128,435,000	$553,836,000	$1,682,271,000

[1] Including gendarmes.

[2] Including Carabinieri.

United States, Japan, and British India				
United States	1910-11	162,357,000	120,729,000	283,086,000
Japan	1911-12	49,196,000	43,405,000	92,601,000
British India	1910-11	100,099,000	....	100,099,000
Totals		311,652,000	$164,134,000	$475,786,000
Mexico and South America				
Argentina	1911	10,583,000	8,236,000	18,819,000
Brazil	1911	24,520,000	20,431,000	44,951,000
Chile	1910	9,852,000	7,653,000	17,505,000
Colombia	1911	1,900,000	....	1,900,000
Ecuador	1909	....	....	1,500,000
Mexico	1911-12	....	....	10,700,000
Peru	1910	....	....	5,400,000
Uruguay	1910	....	....	3,000,000
Venezuela	1910-11	....	....	1,500,000
Totals				$105,275,000
World Total				$2,263,332,000

TABLE C.

EXPENDITURES OF THE TEN CHIEF MILITARY NATIONS.

Country	Fiscal Year	Expended for Army	Expended for Navy	Total military Charge	
France	1911	187,632,000	83,286,000	270,918,000	
Germany	1911-12	203,938,000	114,508,000	318,446,000	Almanach de Gotha, 1911
Great Britain	1910-11	138,800,000	203,020,000	341,820,000	
Italy	1911-12	81,033,000	39,643,000	120,676,000	Statesman's Year-Book, 1911
Japan	1911-12	49,196,000	43,405,000	92,601,000	
Russia	1911	265,642,000	54,128,000	319,770,000	Almanach de Gotha, 1911
Austria-Hungary	1911	$ 73,513,000	$ 13,731,000	$ 87,244,000	
Spain	1911	37,671,000	13,696,000	51,367,000	
Turkey	1911-12	42,071,000	6,223,000	48,294,000	Statesman's Year-Book, 1911
United States	1910-11	162,357,000	120,729,000	283,086,000	U. S. Treas. Report, 1911
Totals		$1,241,853,000	$692,369,000	$1,934,222,000	

TABLE D.

COST OF ARMY PER UNIT OF FIGHTING FORCE FOR THE TEN CHIEF MILITARY NATIONS.

Country	Fighting Force	Cost of Army	Cost per man
Austria-Hungary	396,000	$ 73,513,000	$ 186
France	582,000	187,632,000	322
Germany	626,000	203,938,000	326
Great Britain	[1] 262,000	[1] 138,800,000	[1] 530
Italy	291,000	81,033,000	279
Japan	[2] 225,000	49,196,000	[2] 219
Russia	1,250,000	265,642,000	212
Spain	115,000	37,671,000	328
Turkey	[2] 375,000	[2] 42,071,000	[2] 112
United States	85,000	162,357,000	1910
Totals	4,207,000	$1,241,853,000	$ 295

[1] Regular army only; deducting about $19,600,000 appropriated for Reserves and Territorials, the average, per man of the regular force is about $455.
[2] Uncertain.

TABLE E.

COST OF ARMY AND NAVY PER UNIT OF POPULATION, FOR THE TEN CHIEF MILITARY NATIONS.

Country	[1] Population	Army & Navy	Cost per unit of population.
Austria-Hungary	51,000,000	$ 87,000,000	$1.70
France	39,000,000	271,000,000	7.00
Germany	65,000,000	318,000,000	4.90
Great Britain	45,000,000	342,000,000	7.60
Italy	35,000,000	121,000,000	3.45
Japan	52,000,000	93,000,000	1.79
Russia	160,000,000	320,000,000	2.00
Spain	20,000,000	51,000,000	2.55
Turkey	22,000,000	48,000,000	2.18
United States	92,000,000	283,000,000	3.07
Total	581,000,000	$1,934,000,000	$3.33

[1] World Almanac, 1912.

TABLE F.

PROPORTION OF MILITARY CHARGES TO TOTAL GROSS EXPENDITURES [1] IN TEN LEADING NATIONS.

Country	Total Expenditures	Army & Navy	%
Austria-Hungary [2]	$890,656,000	$ 87,244,000	09.8
France	877,292,000	270,918,000	30.9
Germany	731,286,000	318,446,000	43.5
Great Britain	997,410,000	341,820,000	34.3
Italy	500,595,000	120,676,000	24.1
Japan	284,452,000	92,601,000	32.5
Russia	1,360,054,000	319,770,000	23.5
Spain	224,526,000	51,367,000	22.9
Turkey	154,033,000	48,294,000	31.4
United States	654,138,000	283,086,000	43.3
Totals	$6,674,442,000	$1,934,222.000	29.0

TABLE G.

GROWTH OF EXPENDITURES FOR ARMY, 1881-1911, FOR SEVEN NATIONS.[1]

The estimated total for thirty years is obtained in all cases:

1. By averaging the amounts at the beginning and end of each decade;
2. By averaging the three amounts thus obtained;
3. By multiplying the final average by thirty.

Country	1881	1891	1901
Austria-Hungary [2]	$ 61,827,000	$ 58,645,000	$ 59,726.000
France	113,597,000	141,694,000	138,723,000
Germany	91,075,000	120,964,000	167,588,000
Great Britain	75,126,000	88,640,000	[2]307,500,000
Italy	40,585,000	56,484,000	54,232,000
Russia	90,783,000	123,326,000	162,012,000
United States	38,117,000	44,583,000	134,775,000
Totals	$511,110,000	$634,336,000	$1,024,556,000

[1] Interest on national debt, pension charges and other war matters not belonging to the immediate cost of militarism, are not included under "Army and Navy Expenditures." For example, in the United States the sum of $161,710,367 was paid out for pensions in 1909, and about $21,000,000 as interest on war debt. The total war expenditures for 1909 were thus about $462,000,000, or about 69½ per cent. of the total expenditure. The total expenditure in some nations includes items charged in others to local expenses. (D. S. J.)

[2] This is probably larger than it should be. It is difficult to separate the Imperial expenses from those chargeable to the two separate nations. (A. W. A.)

	1911	Excess over 1881	Estimated total for 30 years
Austria-Hungary ....$	73,513,000	$ 11,686,000	$ 1,860,410,000
France	187,632,000	74,035,000	4,310,315,000
Germany	203,938,000	112,863,000	4,360,585,000
Great Britain ..	138,800,000	63,674,000	3,031,030,000
Italy	81,033,000	40,448,000	1,715,250,000
Russia	265,642,000	174,859,000	4,635,505,000
United States..	162,357,000	124,240,000	2,295,950,000
Totals .....$	1,112,915,000	$601,805,000	$22,209,045,000

TABLE H.

GROWTH OF EXPENDITURES FOR NAVY, 1881-1911, FOR SEVEN NATIONS.

Country	1881	1891	1901
Austria-Hungary ..$	4,355,000	$ 5,672,000	$ 8,698,000
France	42,557,000	43,754,000	65,857,000
Germany	11,434,000	23,470,000	38,195,000
Great Britain	51,130,000	68,935,000	137,615,000
Italy	8,870,000	24,293,000	24,477,000
Russia	13,098,000	21,880,000	46,799,000
United States	13,537,000	22,006,000	55,953,000
Totals ....$	144,981,000	$210,010,000	$377,594,000

	1911	Excess over 1881	Estimated total for 30 years
Austria-Hungary..$	13,731,000	$ 9,376,000	$ 234,130,000
France	83,286,000	40,729,000	1,725,325,000
Germany	114,508,000	103,074,000	1,246,360,000
Great Britain	203,020,000	151,890,000	3,336,250,000
Italy	39,643,000	30,773,000	730,265,000
Russia	54,128,000	41,030,000	1,022,920,000
United States	120,729,000	107,192,000	1,450,920,000
Totals ....$	629,045,000	$484,064,000	$9,746,170,000

[1] These are the only nations that present a fair basis of comparison since 1881.

[2] Reckoned as $107,500,000 in estimating total for 30 years, to allow for extraordinary expenditures in Boer War.

TABLE I.

GROWTH OF COMBINED EXPENDITURES FOR ARMY AND NAVY, 1881-1911, FOR SEVEN NATIONS.

Country	1881	1891	1901
Austria-Hungary..	$ 66,182,000	$ 64,317,000	$ 68,424,000
France	156,154,000	185,448,000	204,580,000
Germany	102,509,000	144,434,000	205,783,000
Great Britain	126,256,000	157,575,000	445,115,000
Italy	49,455,000	80,777,000	78,709,000
Russia	103,881,000	145,206,000	208,811,000
United States	51,654,000	66,589,000	190,728,000
Totals	$656,091,000	$844,346,000	$1,402,150,000

	1911	Excess over 1881	Estimated total for 30 years
Austria-Hungary	$ 87,244,000	$ 21,062,000	$ 2,094,540,000
France	270,918,000	114,764,000	6,035,640,000
Germany	318,446,000	215,937,000	5,606,945,000
Great Britain.	341,820,000	215,564,000	6,367,280,000
Italy	120,676,000	71,221,000	2,445,515,000
Russia	319,770,000	215,889,000	5,658,425,000
United States.	283,086,000	231,432,000	3,996,870,000
Totals ...	$1,741,960,000	$1,085,869,000	$32,205,215,000

TABLE J.

[1]THE GROWTH OF DEBT, 1881-1911, OF THE FIVE GREAT MILITARY NATIONS OF EUROPE.

Country	1891	1891	1901
[2] Austria-Hungary	$1,607,800,000	$ 2,914,876,000	$ 3,219,830,000
France	3,972,407,000	6,400,000,000	6,011,079,000
[3] Germany ..	43,804,000	308,377,000	555,738,000
Italy	1,746,921,000	2,248,200,000	2,451,000,000
Russia	1,225,000,000	1,797,365,000	3,112,000,000
Totals ..	$8,595,932,000	$13,668,818,000	$15,349,647,000

[1] Interest bearing debt only. Issues of paper money are not included.

[2] Austrian Empire, Austria proper and Hungary proper combined. Since 1867 no loans have been contracted by the Empire.

[3] German Empire only. Prussia alone has a separate debt of nearly $2,400,000,000.

	1911	Excess over 1881
[2] Austria-Hungary	$3,612,389,000	$2,004,589,000
France	6,286,435,000	2,314,028,000
[3] Germany	1,224,158,000	1,180,354,000
Italy	2,614,183,000	867,262,000
Russia	4,507,071,000	3,282,071,000

TABLE K.

GROWTH OF INTEREST CHARGE, 1881-1911, OF THE FIVE GREAT MILITARY NATIONS OF EUROPE.

Country	1881	1891	1901
Austria-Hungary	$ 65,108,000	$116,595,000	$128,793,000
France	149,681,000	256,000,000	249,073,000
Germany	1,752,000	12,335,000	18,525,000
Italy	69,900,000	89,818,000	96,000,000
Russia	55,125,000	90,881,000	140,065,000
Totals	$341,566,000	$555,629,000	$632,456,000

	1911	Excess over 1881	Estimated total for 30 years
Austria-Hungary	[1] $144,496,000	$ 79,388,000	$ 3,501,900,000
France	192,762,000	43,081,000	6,762,945,000
Germany	41,981,000	40,229,000	527,265,000
Italy	92,145,000	22,245,000	2,668,405,000
Russia	[1] 180,283,000	125,158,000	3,386,500,000
Totals	$651,667,000	$310,101,000	$16,847,015,000

[1] Estimated at 4%.

[1] TABLE L.

THE FIVE GREAT MILITARY NATIONS OF EUROPE; COMBINED COST OF ARMIES AND NAVIES WITH INCREASE OF INTEREST CHARGES DURING THIRTY YEARS.

Country	Armies and Navies	Increase of interest charges due to increased debt.	Total.
Austria-Hungary	2,094,540,000	$1,548,660,000	$ 3,643,200,000
France	6,035,640,000	2,272,515,000	8,308,155,000
Germany	5,606,945,000	474,705,000	6,081,650,000
Italy	2,445,515,000	571,405,000	3,016,920,000
Russia	5,658,425,000	1,732,750,000	7,391,175,000
Totals	21,841,065,000	$6,600,035,000	$28,441,100,000

[1] See Tables VI to X inclusive.

TABLE M.

APPROPRIATIONS OF THE UNITED STATES.

	Military	Revolutionary pensions	Indian dept.	Naval
1791	$ 947,166,96	$ 280,443.32	$ 40,000.00	$ 33,327.00
1792	1,118,527,91	87,463.60	....	2,000.00
1793	1,068,376,52	82,245,32	100,000.00	....
1794	4,090,669,25	80,239,55	12,942.77	768,888.82
1795	1,063,121.29	85,357.04	50,500.00	....
1796	1,139,614.00	114,259.00	229,000.00	5,000.00
1797	1,440,641.20	96,350.00	19,000.00	487,000.00
1798	4,051,730,95	102,067.07	115,880.00	2,024,712.00
1799	3,243,649.00	93,400.00	207,500.00	3,823,789.89
1800	3,272,020.35	93,000.00	69,500.00	2,482,953.49
Totals to 1815	$109,691,304.22	$2,569,824.90	$4,150,160.53	$56,228,760.78

Total appropriations from 1789 to 1815 .... $404,633,083.03

EXPENDITURES OF THE UNITED STATES.

Expenditures from March 4, 1789, to December 31, 1791 .... $632,804.03

	Military	Pensions	Indian Dept.	Naval.
1791	$ 632,804.03	$ 175,813.88	$ 27,000.00	$ 570.00
1792	1,100,702.09	109,243.15	13,648.85	53.02
1793	1,130,249.08	80,087.81	27,282.83	......
1794	2,639,097.59	81,399.24	13,042.46	61,408.97
1795	2,480,910.13	68,673.22	23,475.68	410,562.03
1796	1,260,263.84	100,843.71	113,563.98	274,784.04
1797	1,039,402.66	92,256.97	62,496.38	382,631.89
1798	2,009,522.30	104,845.33	16,470.09	1,381,347.76
1799	2,466,946.98	95,444.03	20,302.19	2,858,081.84
1800	2,560,878.77	64,130.73	31,031.22	3,448,716.03
Totals to 1815	$102,274,843.19	$2,176,346.90	$3,256,865.66	$53,528,303.93

Total expenditures from 1789 to 1815 .... $378,447,443.69

Figures were taken from the American State Papers—Finance, Vol. II, pp. 920-21.

TABLE N.

(From the Boston Advertiser)

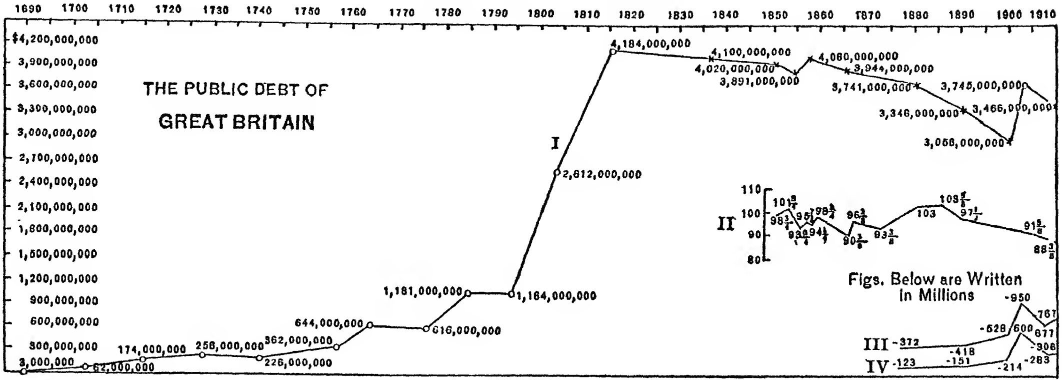
EXPENDITURES OF THE PUBLIC MONEY IN THE UNITED STATES.

	Military	Civil
1899	$201,514,672	$17,371,779
1900	110,175,389	20,767,628
1901	120,070,834	21,009,985
1902	93,974,727	16,097,725
1903	91,591,533	25,890,167
1904	89,010,039	24,752,916
1905	94,119,947	25,317,532
1906	85,962,396	26,693,955
1907	93,525,946	26,040,132
1908	100,431,384	31,293,690
1909	118,204,788	35,691,467
1910	118,953,603	29,740,612
1911	116,741,705	34,558,960

In round numbers since the Spanish War the War Department has spent more than $1,500,000,000, while the operation of the civil government has cost only about $350,000,000.

THE PUBLIC DEBT OF GREAT BRITAIN.

(Prepared by C. R. Nunan)



I. The public debt of Great Britain. [o. Statesman's Year Book, 1895. x. British Sessional Papers, 1909, Vol. L. Cd. 4657.]

II. Fluctuation of British consols. [Source—Journal de la Société de Statistique de Paris (1909), Vol. 50, p. 365.]

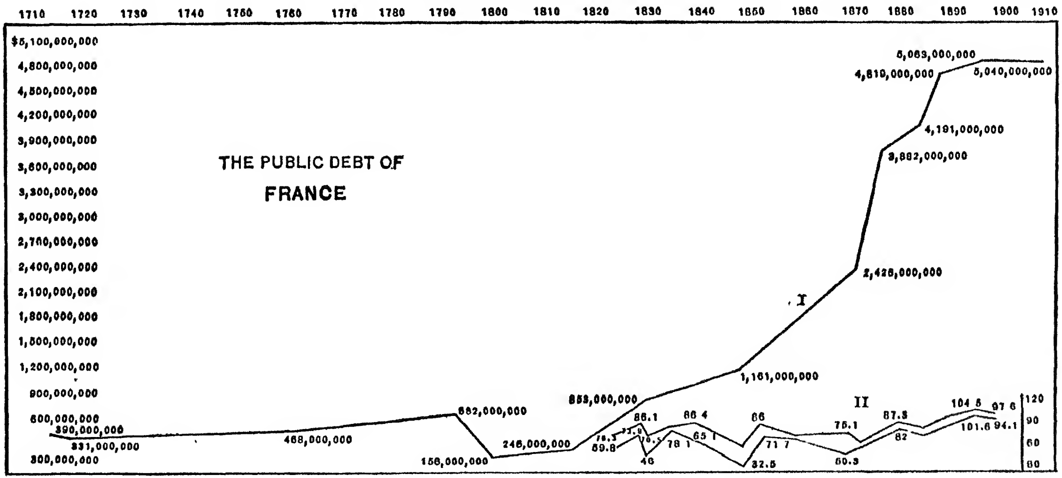
III. Total annual expenditure. [Source—Statesman's Year Book.]

IV. Total annual expenditure for defense. [Source—Statesman's Year Book.]

(The figures for any given year are for the fiscal year ending March 31.)

THE PUBLIC DEBT OF FRANCE.

(*Prepared by L. L. Hill*)



I. The public debt of France. [Source—British Statistical Abstract, 1908, 323.]

II. Fluctation of French bonds (showing both high and low prices). [Source—Journal de la Société de Statistique de Paris, Vol. 50, p. 369.]

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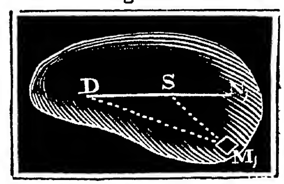
$g = 0{,}031 \left(24 + \frac{25}{9} . 360\right) = 0{,}031 . 1024 = 31{,}75$ lbs.; and hence the accelerated motion of the weight : $p = \frac{24}{31{,}75} = 0{,}756$ feet; on the other hand the acceleration of the motion of the mass $Q, = \frac{b}{a} . p = \frac{25}{15} p = \frac{5 . 0{,}756}{3} = 1{,}26$ ft., and the angular acceleration $= \frac{p}{a} = 0 . 504$ feet. After 4 seconds the acquired angular velocity will be $\omega = 0 . 504 . 4 = 2{,}016$ feet, and the corresponding distance $= \frac{2{,}016 . 4}{2} = 4{,}032$ feet; consequently the angle of revolution $\phi^\circ = \frac{4{,}032}{\pi} . 180^\circ = 1{,}28 . 180^\circ = 230^\circ\ 24'$; consequently the space described by the weight $P = \frac{pt^2}{2} = \frac{0{,}756 . 4^2}{2} = 6{,}05$ feet.

§ 217. *Reduction of the Moment of Inertia.*—When the moment of inertia of a body or system of bodies about an axis, passing through the centre of gravity S of the body, is known, the moment of inertia about any other axis running parallel to it, may be easily found. Let Fig. 250 be the first axis of revolution, passing through the centre of gravity S, D the second axis, for which the moment of inertia of the body is to be determined; further, let $SD = e$ be the distance of the two axes, and let $SN_1 = x_1$ and $N_1M_1 = y_1$ the rectangular co-ordinates of a particle of the mass M_1 of the body. Now the moment of inertia of this particle about $D = M_1 . \overline{DM_1}^2 = M_1 (\overline{DN_1}^2 + \overline{N_1M_1}^2) = M_1 [(e + x_1)^2 + y_1^2]$,

Fig. 250.



and about $S = M_1 . \overline{SM_1}^2 = M_1 . (\overline{SN_1}^2 + \overline{N_1M_1}^2) = M_1 (x_1^2 + y_1^2)$; hence the difference of both moments:

$$= M_1 (e^2 + 2\, ex_1 + x_1^2 + y_1^2) - M_1 (x_1^2 + y_1^2) = M_1 e^2 + 2\, M_1 ex.$$

For another particle of the mass M_2, it is $= M_2 e^2 + 2\, M_2\, ex_2$, for a third $= M_3 e^2 + 2\, M_3\, ex_3$, &c., and for all the particles together:

$$= (M_1 + M_2 + M_3 + \ldots) e^2 + 2e (M_1 x_1 + M_2 x_2 + M_3 x_3 + \ldots).$$

But $M_1 + M_2 + \ldots$ is the sum M of all the masses, and $M_1 x_1 + M_2 x_2 + \ldots$ the sum Mx of their statical moments; hence the difference between the moment of inertia T_1 of the whole body about the axis D and the moment of inertia T about S: $T_1 - T = Me^2 + 2\, eMx$. But since, lastly, for every plane passing through the centre of gravity, the sum of the statical moments of the particles on the one side is as great as that of those on the other, the algebraical sum of all the particles therefore $= 0$; we have, also, $Mx = 0$, and, therefore, $T_1 - T = Me^2$; *i. e.* $T_1 = T + Me^2$.

The moment of inertia of a body about an axis not passing through the centre is equivalent to its moment of inertia about an axis running parallel to it through the centre of gravity, increased by the product of the mass of the body and the square of the distance of the two axes.

It is also seen from this that of all the moments of inertia about parallel axes, that one is the least whose axis is a line of gravity of the body.

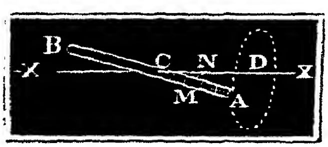
§ 218. It is necessary to know the moments of inertia of the principal geometric bodies, because they very often come into application in mechanical investigations. If these bodies are homogeneous, as in the following we will always suppose to be the case, the particles of the mass M_1, M_2, &c., are proportional to the corresponding particles of the volume V_1, V_2, &c., and hence the measure of the moment of inertia may be replaced by the sum of the particles of the volume, and the squares of their distances from the axis of revolution. In this sense the moments of inertia of lines and surfaces may also be found.

If the whole mass of a body be supposed to be collected into one point, its distance from the axis may be determined on the supposition, that the mass so concentrated possesses the same moment of inertia as if distributed over its space. This distance is called the *radius of gyration, or of inertia.* If T be the moment of inertia, M the mass, and r the radius of gyration, we then have $Mr^2 = T$, and hence $r = \sqrt{\frac{T}{M}}$. We must bear in mind that this radius by no means gives a determinate point, but a circle only, within whose circumference the mass may be considered as arbitrarily distributed.

If into the formula $T_1 = T + Me^2$, we introduce $T = Mr^2$ and $T_1 = Mr_1^2$, we obtain $r_1^2 = r^2 + e^2$, *i. e.* the square of the radius of gyration referred to a given axis = the square of the radius of gyration referred to a parallel line of gravity, plus the square of the distance between the two axes.

§ 219. *The Rod.*—The moment of inertia of a rod AB, Fig. 251, which turns about an axis $X\overline{X}$ through its middle point C, may be determined in the following manner. The cross section of the rod = F, its half length $CA = l$, and the angle which its axis includes with the axis of rotation $ACX = \alpha$. If we divide half the length into n parts, we then obtain n portions each of the contents $\frac{Fl}{n}$; the distances of these portions from the middle C are $\frac{l}{n}, \frac{2l}{n}, \frac{3l}{n}$, &c., hence their distances from the axis $X\overline{X}$ are $MN, = \frac{l}{n} \sin. \alpha\ \frac{2l}{n} \sin. \alpha\ \frac{3l}{n} \sin. \alpha$, &c., and these squares $= \left(\frac{l \sin. \alpha}{n}\right)^2, 4\left(\frac{l \sin. \alpha}{n}\right)^2, 9\left(\frac{l \sin. \alpha}{n}\right)^2$ &c. By the multiplication of these with the contents of an element $\frac{Fl}{n}$, and by the addition of all the products, we have the moment of half the rod:

Fig. 251.



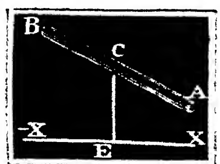
$$T = \frac{Fl}{n}\left[\left(\frac{l \sin. \alpha}{n}\right)^2 + 4\left(\frac{l \sin. \alpha}{n}\right)^2 + 9\left(\frac{l \sin. \alpha}{n}\right)^2 + \dots\right]$$

$= \frac{Fl^3 \sin. \alpha^2}{n^3}(1^2 + 2^2 + 3^2 + \ldots + n^2)$, or, $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n^3}{3}$, $T = \frac{Fl^3 \sin. \alpha^2}{3}$. But as the volume of half the rod Fl is to be considered as the mass M, it follows, finally, that $T = \frac{1}{3} Ml^2 \sin. \alpha^2$. The distance of the end of the rod from the axis $\overline{XX}$ is $AD = a = l \sin. \alpha$, hence it follows more simply that $T = \frac{1}{3} Ma^2$, which formula is also to be applied to the whole rod, if M be the mass of the whole. A mass M_1 at the extremity A of the rod, has the moment of inertia $M_1 a^2$, hence if we make $M_1 = \frac{1}{3} M$, it has then the same moment of inertia as the rod. Whether, therefore, the mass be uniformly distributed over the rod, or its third part be collected at the extremity A, it comes to the same thing.

If we put $T = Mr^2$, we obtain $r^2 = \frac{1}{3} a^2$, and hence the radius of gyration of the rod: $r = a \sqrt{\frac{1}{3}} = 0{,}5773 \,.\, a$.

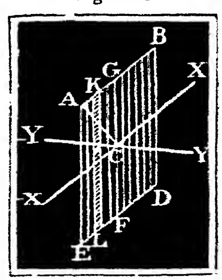
If the rod stands perpendicularly to the axis of rotation $a = l$, therefore, $T = \frac{1}{3} Ml^2$. If, lastly, the rod AB, Fig. 252, be not in the same plane with the axis of rotation, and if the shortest distance between the two axes be $CE = e$, we shall then have from § 217, the moment of inertia $T_1 = T + Me^2 = M(e^2 + \frac{1}{3} a^2)$.

Fig. 252.



§ 220. *Rectangle and Parallelopiped.*—The moment of inertia of a rectangular plate, $ABDE$, Fig. 253, which turns about an axis $\overline{XX}$ passing through its middle C, and parallel to a side, is as for a rod $= \frac{1}{3} Ml^2$, but if the axis $\overline{YY}$ stand perpendicular to the plane, the moment of inertia is then determined from the former paragraph in the following manner; the half $AEFG$ is divided by lines parallel to the side AE into strips of equal breadth, such as KL, the moments of these strips determine, and then added together. If the half length $FE = GA = l$, half the breadth $CF = CG = b$, and the number of parts $= n$, the area of a part $= \frac{l}{n} \,.\, 2\, b = \frac{2\, bl}{n}$. The distances from C of these strips are in the series $\frac{l}{n}, \frac{2l}{n}, \frac{3l}{n}$, &c., therefore their squares $\left(\frac{l}{n}\right)^2, 4\left(\frac{l}{n}\right)^2, 9\left(\frac{l}{n}\right)^2$, &c., hence moments of inertia are:

Fig. 253.



$$\frac{2\, bl}{n}\left[\left(\frac{l}{n}\right)^2 + \frac{b^2}{3}\right], \frac{2\, bl}{n}\left[4\left(\frac{l}{n}\right)^2 + \frac{b^3}{3}\right], \frac{2\, bl}{n}\left[9\left(\frac{l}{n}\right)^2 + \frac{b^2}{3}\right]$$

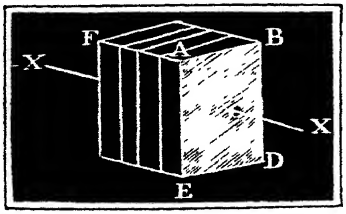
and the moment of inertia of half the plate:

$$T = \frac{2\, bl}{n}\left[\left(\frac{l}{n}\right)^2 (1 + 4 + 9 + \ldots + n^2) + n \,.\, \frac{b^2}{3}\right]$$

$$= \frac{2\,bl}{n}\left[\left(\frac{l}{n}\right)^2 \cdot \frac{n^3}{3} + \frac{n\,b^2}{3}\right] = \frac{2\,bl\,(l^2 + b^2)}{3} = \tfrac{1}{3}\,M\,(l^2 + b^2),$$

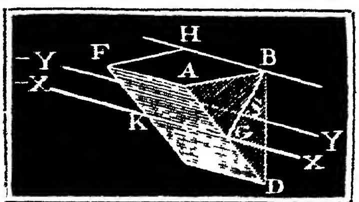
because $2\,bl$ must be considered as the mass of half the plate. As the semi-diagonal $CA = d = \sqrt{l^2 + b^2}$, we may put: $T = \frac{1}{3} Md^2$. If M represent the whole mass, the formula holds good for the moment of inertia of the whole plate. Since, further, a parallelopiped BEF, Fig. 254, may be decomposed by parallel planes into equal rectangular plates, the above formula is also applicable to this, if the axis of rotation pass through the middle points of any two opposite surfaces. It follows, besides, from this formula, that the moment of inertia of the parallelopiped is equivalent to the moment of inertia of the third part of its mass applied at a corner A.

Fig. 254.



From the formula for the moment of inertia of a parallelopiped, that of a triangular prism may be also calculated. The diagonal plane ADF divides the parallelopiped into two equal triangular prisms with rectangular and triangular bases ABD, Fig. 255; hence, for the rotation about an axis $\overline{XX}$, passing through the middle C and K of the hypothenuse, the moment of inertia $= \frac{1}{3}\,Md^2$. If now we make use of the proposition § 217, we obtain the moment of inertia about an axis $\overline{YY}$, passing through the centres of gravity S and S_1 of the bases: T

Fig. 255.



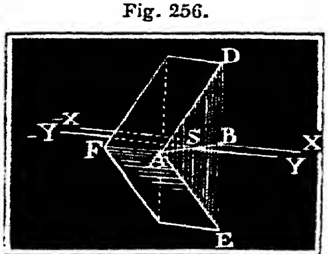
$$= \tfrac{1}{3}\,Md^2 - M\,.\,\overline{CS}^2 = M\left[\frac{d^2}{3} - (\tfrac{1}{3}\,\overline{CB})^2\right] = M\left[\frac{d^2}{3} - \left(\frac{d}{3}\right)^2\right]$$

$= \frac{2}{9}\,Md^2$, and it follows also that the moment of inertia about a side edge BH:

$T_1 = T + M\,.\,\overline{SB}^2 = \frac{2}{9}\,Md^2 + M\,(\frac{2}{3}\,d)^2 = \frac{6}{9}\,Md^2 = \frac{2}{3}\,Md^2$, where d always represents half the hypothenuse of the triangular base.

§ 221. *Prisms and Cylinders.*—For a prism $ADFE$, Fig. 256, with isosceles triangular bases, the moment of inertia about an axis $\overline{XX}$, which connects the middle points of the bases, $T = \frac{2}{3}\,Md^2$, if d represent half the side AD of the surface of the base, because this surface may be decomposed by the line of the height AB into two equal rectangular triangles. If now the height AB of the isosceles triangular base $= h$, we have then for the moment of inertia

Fig. 256.



of this prism about the axis $\overline{YY}$ passing through the centres of gravity of the base:

$$T = \tfrac{2}{3}\, M\, d^2 - M\left(\frac{h}{3}\right)^2 = M\left(\tfrac{2}{3}\, d^2 - \tfrac{1}{9}\, h^2\right) = \tfrac{1}{3}\, M\left(2\, d^2 - \tfrac{1}{3}\, h^2\right),$$

and finally, the moment of inertia about the edge passing through the points A and F of the bases:

$$T_1 = T + M\left(\tfrac{2}{3} h\right)^2 = M\left(\frac{2\, d^2}{3} - \frac{h^2}{9} + \frac{4\, h^2}{9}\right) = \tfrac{1}{3}\, M\left(2\, d^2 + h^2\right).$$

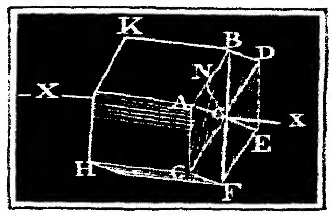
Hence the moment of inertia of a right and regular prism revolving about its geometric axis may be found. Let h be the height CA, Fig. 257, of one of the supplementary triangles, $CA = CB = 2\, d = r$ the radius of the base or of a supplementary triangle, and M the mass of the entire prism. We have then by the last formula:

$$T = \tfrac{1}{3}\, M\left(\frac{r^2}{2} + h^2\right).$$

The regular prism becomes a cylinder when $h = r$, hence the moment of inertia of a right cylinder about its geometric axis is:

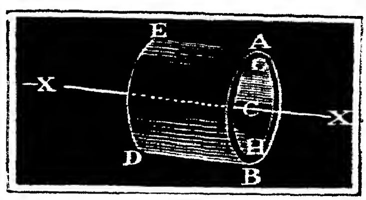
$$T = \tfrac{1}{3}\, M\left(\frac{r^2}{2} + r^2\right) = \tfrac{1}{2}\, Mr^2.$$

Fig. 257.



The moment of inertia of a cylinder is, therefore, equivalent to the moment of inertia of half the mass of the cylinder collected at its circumference, or equivalent to the moment of inertia of the entire mass, at the distance $r\sqrt{\tfrac{1}{2}} = 0{,}7071\,.\, r$.

Fig. 258.



If the cylinder $ABDE$ be hollow, Fig. 258, the moment of inertia of the hollow space must be subtracted from that of the solid cylinder. If the outer radius $CA = r_1$, and the inner $CG = r_2$, we then have from what has preceded the moment of inertia of the hollow cylinder:

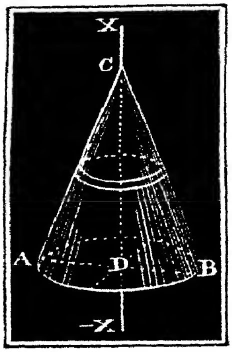
$$T = \tfrac{1}{2}\left(M_1 r_1^2 - M_2 r_2^2\right) = \tfrac{1}{2}\,\pi\left(r_1^2\,.\,r_1^2 - r_2^2\,.\,r_2^2\right) = \tfrac{1}{2}\,\pi\left(r_1^4 - r_2^4\right)$$
$$= \tfrac{1}{2}\,\pi\left(r_1^2 - r_2^2\right)\left(r_1^2 + r_2^2\right) = \tfrac{1}{2}\, M\left(r_1^2 + r_2^2\right),$$

because the volume considered as the mass $= \pi\left(r_1^2 - r_2^2\right)$. If r be the mean radius $\dfrac{r_1 + r_2}{2}$, and b the breadth of the annular surface, we then have $T = M\left(r^2 + \dfrac{b^2}{4}\right)$.

§ 222. *Cone and Sphere.*—The moment of inertia of a right cone, as well as that of a sphere, may be calculated from the formula for the moment of inertia of a cylinder. Let ACB, Fig. 259, be a cone revolving about its geometric axis, $DA = DB = r$ the radius of its base, and $CD = h$ its height coinciding with the axis.

If we make n slices parallel to the base at equal distances, we then obtain thin discs of the radii $\frac{r}{n}, 2\frac{r}{n}, 3\frac{r}{n} \ldots n\frac{r}{n}$, and of the common height $\frac{h}{n}$. The half volumes of these discs are $\pi\left(\frac{r}{n}\right)^2 \cdot \frac{h}{2n}, \pi\left(\frac{2r}{n}\right)^2 \cdot \frac{h}{2n}, \pi \cdot \left(\frac{3r}{n}\right)^2 \cdot \frac{h}{2n}$, &c., and hence their moments of inertia:

Fig. 259.



$$\pi\left(\frac{r}{n}\right)^4 \cdot \frac{h}{2n}, \pi\left(\frac{2r}{n}\right)^4 \frac{h}{2n}, \pi\left(\frac{3r}{n}\right)^4 \frac{h}{2n}, \text{\&c.};$$

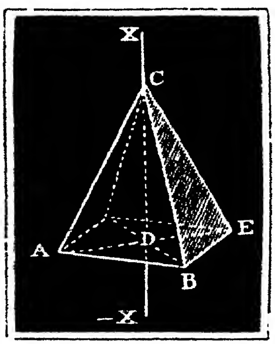
the sum of these values gives, finally, the moment of inertia of the entire cone:

$$T = \frac{\pi r^4 h}{2n^5}\, 1^4 + 2^4 + 3^4 + \ldots + n^4),$$

and as $1^4 + 2^4 + 3^4 + \ldots + n^4 = \frac{n^5}{5}$, we have

$$T = \frac{\pi r^4 h}{10} = \frac{3}{10} \cdot \frac{\pi r^2 h}{3} \cdot r^2 = \frac{3}{10} Mr^2.$$

Fig. 260.



For the right pyramid ACE, Fig. 260, with rectangular base, under the same circumstances $T = \frac{1}{5} Md^2$, if d represent the semi-diagonal DA of the base. Also by subtraction of the two moments of inertia, the moment of inertia of a right truncated cone with the radii r_1 and r_2 and the heights h_1 and h_2 may be obtained:

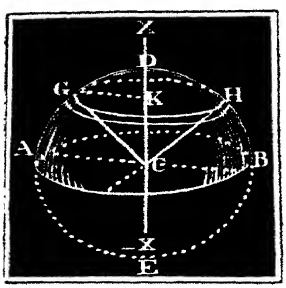
$$T = \frac{\pi}{10}(r_1^4 h_1 - r_2^4 h_2) = \frac{\pi h_1}{10 r_1}(r_1^5 - r_2^5),$$

or, since the mass

$$M = \frac{\pi}{3}(r_1^2 h_1 - r_2^2 h_2) = \frac{\pi h_1}{3 r_1}(r_1^3 - r_2^3),$$

$$T = \tfrac{3}{10} M \left(\frac{r_1^5 - r_2^5}{r_1^3 - r_2^3}\right).$$

Fig. 261.



In a similar manner, we find the moment of inertia of a sphere, revolving about one of its diameters $DE = 2r$. Let us divide the hemisphere ADB, Fig. 261, by sections parallel to the base ACB, into n equally thick circular slices, as GKH, &c., and determine their moments. The square $\overline{GK}^2$ of the radius of any such slice is:

$$= \overline{CG}^2 - \overline{CK}^2 = r^2 - \overline{CK}^2,$$

hence its moment of inertia

$$= \tfrac{1}{2}\pi \cdot \frac{r}{n}(r^2 - \overline{CK}^2)^2,$$

$$= \frac{\pi r}{2n}(r^4 - 2r^2\,\overline{CK}^2 + \overline{CK}^4).$$

Let us put in succession for CK $\frac{r}{n}, \frac{2r}{n}, \frac{3r}{n}$, &c., to $\frac{nr}{n}$, and add the results, we shall then have the moment of inertia of the sphere:

$$T = \frac{\pi}{2n}\left[n \cdot r^4 - 2r^2\left(\frac{r}{n}\right)^2(1^2 + 2^2 + 3^2 + \ldots + n^2) + \left(\frac{r}{n}\right)^4(1^4 + 2^4 + 3^4 + \ldots + n^4)\right] = \frac{\pi r}{2n}\left[nr^4 - \frac{2r^4}{n^2}\cdot\frac{n^3}{3} + \left(\frac{r}{n}\right)^4\cdot\frac{n^5}{5}\right] = \frac{\pi r^5}{2}(1 - \tfrac{2}{3} + \tfrac{1}{5}) = \frac{4\pi r^5}{15}.$$

Now the solid contents of a hemisphere $M = \frac{2}{3}\pi r^3$, hence we may put:

$$T = \tfrac{2}{5}\cdot\tfrac{2}{3}\pi r^3 \cdot r^2 = \tfrac{2}{5} Mr^2,$$

and if we take M for the whole sphere, the formula will hold good for the case.

The formula $T = \frac{2}{5} Mr^2$ is true also for a spheroid whose equatorial radius $= r$ (§ 117).

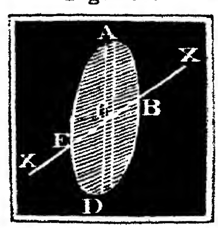
If the sphere revolves about another axis at the distance e from its centre, the moment of inertia is then

$$T = M(e^2 + \tfrac{2}{5} r^2).$$

The radius of gyration $= r\sqrt{\frac{2}{5}} = 0{,}6324 \,.\, r$; two-fifths of the mass of the sphere at a distance from the axis of rotation equal to the radius of the sphere, have the same moment of inertia as the whole sphere.

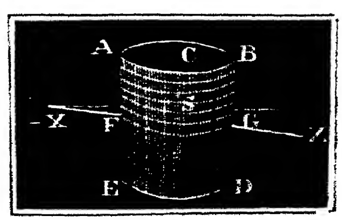
§ 223. The moment of inertia of a circular disc $ABDE$, Fig. 262, revolving about its diameter BE, is, as for the moment of flexure of a cylinder, (§ 195) $= \frac{\pi r^4}{4} = \frac{Mr^2}{4}$, consequently the radius of gyration $= r\sqrt{\frac{1}{4}} = \frac{1}{2} r$, *i. e.* half the radius of the circle.

Fig. 262.



From this we may now find the moment of inertia of a cylinder $ABDE$, Fig. 263, which revolves about a diameter FG, passing through the centre of gravity S. If l be half the height and r the radius of the cylinder, we then have the volume of half the cylinder $= \pi r^2 l$, and if we make equi-distant sections parallel to the base, we decompose this body into n equal parts, each of which $= \frac{\pi r^2 l}{n}$, the first is distant $\frac{l}{n}$ from the centre of gravity S, the second $\frac{2l}{n}$, the third $\frac{3l}{n}$, &c. In virtue of the formula, § 217, the moments of inertia of these circular slices are:

Fig. 263.



$$\frac{\pi r^2 l}{n}\left[\tfrac{1}{4}r^2+\left(\frac{l}{n}\right)^2\right],\ \frac{\pi r^2 l}{n}\left[\tfrac{1}{4}r^2+\left(\frac{2l}{n}\right)^2\right],$$
$$\frac{\pi r^2 l}{n}\left[\tfrac{1}{4}r^2+\left(\frac{3l}{n}\right)^2\right],$$

&c., their sum gives the moment of inertia of half the cylinder:

$$T=\frac{\pi r^2 l}{n}\left[\frac{nr^2}{4}+\left(\frac{l}{n}\right)^2(1^2+2^2+3^2+\ldots+n^2)\right]=$$
$$\pi r^2 l\left(\frac{r^2}{4}+\frac{l^2}{n^3}\cdot\frac{n^3}{3}\right)=M\left(\frac{r^2}{4}+\frac{l^2}{3}\right)$$

which holds good likewise for the whole cylinder, if M represents its mass.

We find in like manner for the right cone ABD, Fig. 264, whose axis of revolution passes through its centre of gravity, and is perpendicular to the geometrical axis CD

$$T=\tfrac{3}{20}M\left(r^2+\frac{h^2}{4}\right).$$

Fig. 264.

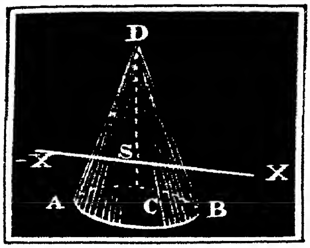
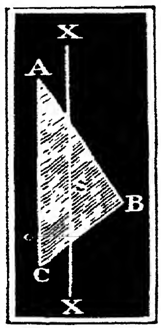


Fig. 265.



For a plate ABC, Fig. 265, in the form of a rectangular triangle, the moment of inertia about an axis passing through the centre of gravity S, and parallel to the side AC, is according to § 193:

$$T=\frac{bh^3}{36}=\frac{bh}{2}\cdot\frac{h^2}{18}=\tfrac{1}{18}Mh^2,$$

if the breadth b parallel to the axis of revolution, and h the height perpendicular to it be given. This formula holds good even for an oblique angled triangle, if the axis runs parallel to the base, and h represents the height of the triangle. From this the moment of inertia of a triangular prism $ADEF$, Fig. 266, may be found, if the axis of revolution $X\overline{X}$ passes through its centre of gravity S, and is parallel to the side DE of the surface of the base, it follows from the same method as that adopted for the cylinder, that

$$T=M\left(\tfrac{1}{18}h^2+\frac{l^2}{3}\right),$$

where l represents half the length of the prism.

Fig. 266.

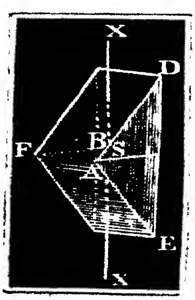
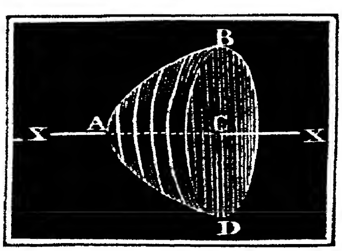


Fig. 267.



§ 224. *Segments.*—The moment of inertia of a paraboloid of revolution, BAD, Fig. 267, which turns about its axis of rotation AC, is

determined in a similar manner to that of a sphere. Let the radius of the base $CB = CD = a$, the height $CA = h$, and the body consist of n slices, each of the height $\frac{h}{n}$, we have then the contents of these

$$= \frac{h}{n} \cdot \pi \cdot \frac{1}{n} a^2, \frac{h}{n} \pi \cdot \frac{2}{n} a^2, \frac{h}{n} \cdot \pi \frac{3}{n} a^2, \text{\&c.},$$

because the squares of the radii are as the heights. From this the moments of inertia are given

$$= \frac{h}{n} \cdot \frac{\pi}{2} \cdot \frac{a^4}{n^2}, \frac{h}{n} \cdot \frac{\pi}{2} \cdot \frac{4\, a^4}{n^2}, \frac{h}{n} \cdot \frac{\pi}{2} \cdot \frac{9\, a^4}{n^2}, \text{\&c.},$$

and hence, finally, it follows that the moment of inertia of the whole paraboloid is

$$T = \frac{\pi a^4 h}{2n^3}(1^2+2^2+3^2+\ldots+n^2) = \frac{\pi a^4 h}{2n^3} \cdot \frac{n^3}{3} = \frac{\pi a^4 h}{6}$$

$$= \frac{\pi a^2 h}{2} \cdot \frac{a^2}{3} = \tfrac{1}{3} Ma^2,$$

because the volume of this body is $M = \frac{\pi a^2 h}{2}$.

The same formula is applicable to a small segment of a sphere, but if the height h is not very small compared with a, we have to put the moment of inertia of slices

$$= \frac{\pi h}{2n} \cdot a^4 = \frac{\pi h}{2n} \cdot h^2 (2\, r - h)^2 = \frac{\pi h}{2n} \cdot (4\, r^2\, h^2 - 4\, rh^3 + h^4),$$

where r represents the radius of the sphere. If now we take successively for h the values $\frac{h}{n}, \frac{2h}{n}, \frac{3h}{n}$, &c., we then obtain for the moment of inertia of a segment of a sphere

$$T = \frac{\pi h}{2n}\left[4r^2\left(\frac{h}{n}\right)^2 \cdot \frac{n^3}{3} - 4r\left(\frac{h}{n}\right)^3 \cdot \frac{n^4}{4} + \left(\frac{h}{n}\right)^4 \cdot \frac{n^5}{5}\right]$$

$$= \frac{\pi h^3}{30}(20\, r^2 - 15\, rh + 3\, h^2).$$

The contents of the segment are $M = \pi h^2 (r - \frac{1}{3} h)$, hence

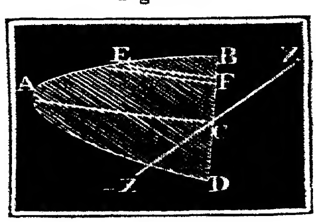
$$T = \pi h^2 (r - \tfrac{1}{3} h) \cdot \frac{2h}{3}\left(r - \tfrac{5}{12} h + \tfrac{1}{90} \cdot \frac{h^2}{r - \frac{1}{3} h}\right)$$

$$= \tfrac{2}{3} Mh \left(r - \tfrac{5}{12} h + \tfrac{1}{90} \cdot \frac{h^2}{r - \frac{1}{3} h}\right).$$

Generally $T = \frac{2}{3} Mh (r - \frac{5}{12} h)$ is sufficiently correct. This formula finds its application in pendulum-bobs.

The moment of inertia of the surface of a parabola ABD, Fig. 268, which revolves about an axis $X\overline{X}$, passing through the middle C of the chord BD, is found if the surface be decomposed into equally broad stripes, such as EF, and their moments added together. Let $AC = l$ be the length, and $CB = b$ half the breadth of the sur-

Fig. 268.



face, $CF = x$ the absciss, and $EF = y$ the ordinate or length of an element. Its moment of inertia is then $= \frac{b}{n} y \left(x^2 + \frac{y^2}{3}\right)$; but as $\frac{x^2}{b^2} = \frac{l - y}{l}$, therefore $y = l\left(1 - \frac{x^2}{b^2}\right)$, it follows that this moment $= \frac{b}{n}\left[lx^2\left(1 - \frac{x^2}{b^2}\right) + l^3\left(1 - \frac{x^2}{b^2}\right)^3\right]$. If now x be successively put $= \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}$, &c., and the results added, we obtain the moment of inertia of half the surface of the parabola:

$$T = bl\left[\frac{l^2}{3n}\left(n - n + \tfrac{3}{5}\, n - \frac{n}{7}\right) + \frac{b^2}{3} - \frac{b^2}{5}\right]$$

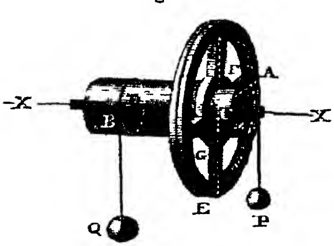
$$= bl\left(\frac{16\, l^2}{3\,.\,35} + \frac{2\, b^2}{3\,.\,5}\right) = \tfrac{2}{3}\, bl\left(\tfrac{8}{35}\, l^2 + \tfrac{1}{5}\, b^2\right) = \tfrac{1}{5}\, M\left(\tfrac{8}{7}\, l^2 + b^2\right),$$

because the surface of the parabola is $M = \frac{2}{3}\, bl$.

This formula, which holds good for the entire surface of the parabola, is also applicable to a prism having a parabolic surface for the base, as in vibrating beams.

§ 225. *Wheel and Axle.*—The theory of the moment of inertia finds its most frequent application in machines and instruments, because in these rotatory motions about a fixed axis are those which generally present themselves. Many applications of this doctrine will be met with in the sequel, hence it will suffice to treat of only a few simple cases for the present.

Fig. 269.



If two weights P and Q, act on a wheel and axle $ACDB$, Fig. 269, with the arms $CA = a$ and $DB = b$ through the medium of a perfectly flexible string, and if the radius of the gudgeons be so small that their friction may be neglected, it will remain in equilibrium if the statical moments $P\,.\,CA$ and $Q\,.\,DB$ are equal, and therefore $Pa = Qb$. But if the moment of the weight P is greater than that of Q, therefore, $Pa > Qb$, P will descend and Q ascend; if $Pa < Qb$, P will ascend and Q descend. Let us now examine the conditions of motion in one of the latter cases. Let us suppose that $Pa > Qb$. The force corresponding to the weight Q and acting at the arm b generates at the arm a a force $\frac{Qb}{a}$, which acts opposite to the force corresponding to the weight P, and hence there is a residuary moving force $P - \frac{Qb}{a}$ acting at A. The mass $\frac{Q}{g}$ is reduced by its transference from the distance b to that of a to $\frac{Qb^2}{ga^2}$, hence the mass moved by $P - \frac{Qb}{a}$ is

$M = \left(P + \frac{Qb^2}{a^2}\right) \div g$, or, if the moment of inertia of the wheel and axle $= \frac{Gy^2}{g}$, and, therefore, its inert mass reduced to $A = \frac{Gy^2}{ga^2}$, we have more exactly:

$$M = \left(P + \frac{Qb^2}{a^2} + \frac{Gy^2}{a^2}\right) \div g = (Pa^2 + Qb^2 + Gy^2) \div ga^2.$$

From thence it follows that the accelerated motion of the weight P, together with that of the circumference of the wheel, namely

$$p = \frac{\text{moving force}}{\text{mass}} = \frac{P - \frac{Qb}{a}}{Pa^2 + Qb^2 + Gy^2} \cdot ga^2$$
$$= \frac{Pa - Qb}{Pa^2 + Qb^2 + Gy^2} \cdot ga;$$

on the other hand, the accelerated motion of the ascending weight Q, or of the circumference of the wheel is:

$$q = \frac{b}{a} p = \frac{Pa - Qb}{Pa^2 + Qb^2 + Gy^2} \cdot gb.$$

The tension of the string by P is $S = P - \frac{Pp}{g} = P\left(1 - \frac{p}{g}\right)$ (§ 73), that of the string by Q : $T = Q + \frac{Qq}{g} = Q\left(1 + \frac{q}{g}\right)$ hence the pressure on the gudgeon is:

$$S + T = P + Q - \frac{Pp}{g} + \frac{Qq}{g} = P + Q - \frac{(Pa - Qb)^2}{Pa^2 + Qb^2 + Gy^2};$$

the pressure, therefore, on the gudgeons for a revolving wheel and axle is here less than in a state of equilibrium. Lastly, from the accelerating forces p and q, the rest of the relations of motion may be found. After t seconds the velocity of P is $v = pt$, of Q : $v_1 = qt$, and the space described by P : $s = \frac{1}{2} pt^2$, by Q : $s_1 = \frac{1}{2} qt^2$.

Example. Let the weight P at the wheel be $= 60$ lbs., that at the axle $Q = 160$ lbs., the arm of the first $CA = a = 20$ inches, that of the second $DB = b = 6$ inches; further, let the axle consist of a solid cylinder of 10 lbs. weight, and the wheel of two iron rings and four arms, the rings of 40 and 12 lbs., the arms, together, of 15 lbs. weight; lastly, let the radii of the greater ring $AE = 20$ and 19 inches, that of the less $FG = 8$ and 6 inches. Required, the conditions of motion of this machine. The moving force at the circumference of the wheel is:

$$P - \frac{b}{a} Q = 60 - \frac{6}{20} \cdot 160 = 60 - 48 = 12 \text{ lbs.},$$

the moment of inertia of the machine, neglecting the masses of the gudgeons and the strings, is equivalent to the moment of inertia of the axle $= \frac{Wb^2}{2} = \frac{10 \cdot 6^2}{2} = 180$, plus the moment of the smaller ring $= \frac{R_1(r_1^2 + r_2^2)}{2} = \frac{12 \cdot (8^2 + 6^2)}{2} = 600$, plus the moment of the larger ring $= \frac{40 \cdot (20^2 + 19^2)}{2} = 15220$, plus the moment of the arms, approximately $= \frac{A(\rho_1^3 - \rho_2^3)}{3(\rho_1 - \rho_2)} = \frac{A(\rho_1^2 + \rho_1\rho_2 + \rho_2^2)}{3} = \frac{15 \cdot (19^2 + 19 \cdot 8 + 8^2)}{3} = 2885$; hence, collectively, $Gy^2 = 180 + 600 + 15220 + 2885 = 18885$, or for foot measure

$= \frac{18885}{144} = 131{,}14$. The collective mass, reduced to the circumference of the wheel is:

$$= \left(P + \frac{Qb^2 + Gy^2}{a^2}\right) \div g = \left[60 + 160\left(\frac{6}{20}\right)^2 + \frac{18885}{20^2}\right] \div g$$

$= \left(60 + 160 \,.\, 0{,}09 + \frac{18885}{400}\right) . \,0{,}031 = 121{,}61 \times 0{,}031 = 377$ lbs.

Accordingly, the accelerated motion of the weight P, together with that of the circumference of the wheel, is:

$= \frac{P - \frac{b}{a} Q}{P + \frac{Qb^2 + Gy^2}{a^2}} \,.\, g = \frac{12}{3{,}77} = 3{,}183$ feet; on the other hand, that of $Q : q = \frac{b}{a} p$

$= \frac{6}{20} \,.\, 3{,}183 = 0{,}954$ feet; further, the tension of the string by P is $= \left(1 - \frac{p}{g}\right) P$

$= \left(1 - \frac{3{,}133}{32{,}2}\right) . \, 60 = 54{,}07$ lbs.; that by Q, on the other hand, $Q = \left(1 + \frac{q}{g}\right) . \, Q$

$= (1 + 0{,}925 \,.\, 0{,}031) \,.\, 160 = 1{,}030 \,.\, 160 = 164{,}8$ lbs.; and consequently the pressure on the gudgeons $S + T = 54{,}06 + 164{,}80 = 218{,}86$ lbs., or inclusive of the weight of the machine $= 218{,}86 + 77 = 295{,}86$ lbs. After 10 seconds, P has acquired the velocity $pt = 3{,}084 \,.\, 10 = 30{,}84$ feet, and described the space $s = \frac{vt}{2} = 30{,}84 \,.\, 5$

$= 154{,}2$ feet, and Q has ascended a height $\frac{b}{a} s = 0{,}3 \,.\, 154{,}2 = 46{,}26$ feet.

§ 226. The weight P which communicates to the weight Q the accelerated motion $q = \frac{Pab - Qb^2}{Pa^2 + Qb^2 + Gy^2} \,.\, g$ may also be replaced by another weight P_1, without changing the acceleration of the motion Q, if it act at the arm a, for which:

$$\frac{P_1 a_1 - Qb}{P_1 a_1^2 + Qb^2 + Gy^2} = \frac{Pa - Qb}{Pa^2 + Qb^2 + Gy^2}.$$

The magnitude $\frac{Pa^2 + Qb^2 + Gy^2}{Pa - Qb}$, represented by k, and we obtain a_1^2

$- ka_1 = - \frac{Qb\,(b + k) + Gy^2}{P_1}$, and the arm in question:

$$a_1 = \tfrac{1}{2} k \pm \sqrt{\left(\frac{k}{2}\right)^2 - \frac{Qb\,(b+k) + Gy^2}{P_1}}.$$

We may also find by help of the differential calculus, that the motion of Q is most accelerated by the weight P, when the arm of the latter corresponds to the equation $Pa^2 - 2\,Qab = Qb^2 + Gy^2$, therefore,

$$a = \frac{b\,Q}{P} + \sqrt{\left(\frac{b\,Q}{P}\right)^2 + \frac{Qb^2 + Gy^2}{P}}.$$

The formula found above assumes a complicated form if the friction of the gudgeons and the rigidity of the cord are taken into account. If we represent the statical moments of both resistances by Fr, we must then substitute for the moving force $P - \frac{b}{a} Q$, the value P

$- \frac{Qb + Fr}{a}$, whence the acceleration of Q comes out

$$q = \frac{(Pa - Fr)\, b - Qb^2}{Pa^2 + Qb^2 + Gy^2} \cdot g \text{ and}$$
$$a = \frac{Qb + Fr}{P} + \sqrt{\left(\frac{Qb + Fr}{P}\right)^2 + \frac{Qb^2 + Gy^2}{P}}.$$

Examples.—1. The weights $P = 30$ lbs. $Q = 80$ lbs. act at the arms $a = 2$ feet, and $b = \frac{1}{2}$ foot of a wheel and axle, and their moments of inertia Gy^2 amount to 60 lbs.; then the accelerated motion of the ascending weight Q is:
$q = \frac{30 \,.\, 2 \,.\, \frac{1}{2} - 80 \,.\, (\frac{1}{2})^2}{30 \,.\, 2^2 + 80 \,.\, (\frac{1}{2})^2 + 60} \cdot g = \frac{30 - 20}{120 + 20 + 60} \,.\, 32{,}2 = \frac{322}{200} = 1{,}61$ feet. But if a weight $P_1 = 45$ lbs. generates the same acceleration in the motion of Q, the arm of P_1 is then:
$a_1 = \frac{k}{2} \pm \sqrt{\left(\frac{k}{2}\right)^2 - \frac{80 \,.\, \frac{1}{2}\,(\frac{1}{2} + k) + 60}{45}}$, or as $k = \frac{200}{60 - 40} = 10$, a_1 is $= 5 \pm \sqrt{25 - \frac{32}{3}} = 5 \pm \frac{1}{3}\, 11{,}358 = 5 \pm 3{,}786 = 8{,}786$ feet, or $1{,}214$ feet.—2. The accelerated motion of Q comes out greatest if the arm of the force or radius of the wheel amount to:
$a = \frac{\frac{1}{2} \,.\, 80}{30} + \sqrt{\left(\frac{40}{30}\right)^2 + \frac{20 + 60}{30}} = \frac{4}{3} + \sqrt{\frac{16}{9} + \frac{24}{9}} = \frac{4 + \sqrt{40}}{3} = 3{,}4415$ feet, and q is $= \left(\frac{30 \,.\, 1{,}7207 - 20}{30 \,.\, (3{,}4415)^2 + 80}\right) g = \frac{31{,}621}{435{,}32} \cdot g = 2{,}339$ feet.—3. The statical moment of the friction, together with the rigidity of the string, is $Fr = 8$; then, instead of Qb, we must put $Qb + Fr = 40 + 8 = 48$; whence it follows that: $a = \frac{48}{30} + \sqrt{\left(\frac{40}{30}\right)^2 + \frac{8}{3}} = 1{,}6 + \sqrt{5{,}227} = 3{,}886$, and the correspondent maximum accelerating force $q = \frac{30 \,.\, 1{,}943 - 8 \,.\, \frac{1}{2} - 20}{30 \,.\, (3{,}886)^2 + 80} \cdot g = \frac{34{,}29}{533} \,.\, 32 \,.\, 2 = 2{,}071$ feet.

§ 227. *Attwood's Machine.*—The formulæ found in § 225 for the wheel and axle hold good also for the simple fixed pulley, for if $b = a$, the wheel and axle becomes either a pulley or an axle. Retaining the other denominations of the paragraph mentioned, we have then for the accelerating motion with which P descends and Q ascends: $p = q = \frac{(P - Q)\, a^2}{(P + Q)\, a^2 + Gy^2} \cdot g$, or having regard to friction:

$$p = q = \frac{(P - Q)\, a^2 - Far}{(P + Q)\, a^2 + Gy^2} \cdot g.$$

In order to diminish the friction of the gudgeons, the gudgeon C of the pulley AB, Fig. 270, is placed upon the friction wheels DEF and $D_1E_1F_1$. The moments of inertia of these are $G_1y_1^2$, and their radii $DE = D_1E_1 = a_1$, we then have to put:

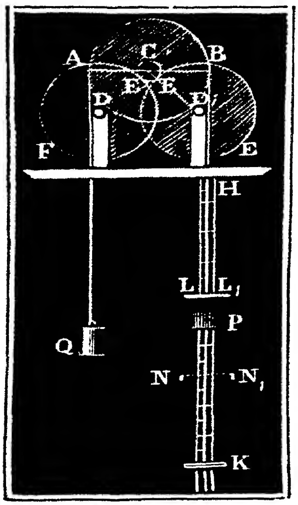
$$p = q = \frac{(P - Q)\, a^2 - Far}{(P + Q)\, a^2 + Gy^2 + G_1 \dfrac{y_1^2 r^2}{a_1^2}} \cdot g,$$

because the inert masses of these wheels reduced to the circumference of the friction wheels, or to the gudgeon of the wheel, $= \frac{G_1 y_1^2}{a_1^2}$. By inversion we obtain the accelerating force of gravity:

$$g = \frac{(P+Q)\,a^2 + Gy^2 + G_1\,\dfrac{y_1^{\,2} r^2}{a_1^{\,2}}}{(P-Q)\,a^2 - Far} \cdot p.$$

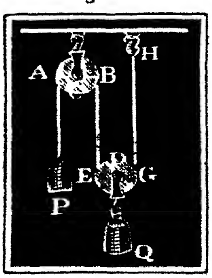
For a small difference of the two weights $P - Q$ the accelerating force p comes out small, hence the motion goes on slowly, and if the resistance opposed to the weight by the air be inconsiderable, with the assistance of experiments upon the descent of weights in such an arrangement, the accelerating force of gravity may be measured with tolerable accuracy, which by a body falling freely it is impossible to do. Experiments of this kind were first instituted by Attwood,* whence this arrangement is known by the name of Attwood's machine. To determine the spaces fallen through, there is a scale HK along which the weight P descends. From the space fallen through s, and the corresponding time t, it follows of course that $p = \frac{2 \cdot s}{t^2}$; if, however, during the descent, the moving force be removed, while an equal weight LL_1 forming a hollow ring is taken up by a fixed narrower ring NN_1, the remaining part of the space s_1 will be described with a uniform motion, and having the time observed by a good clock, the velocity will be given $v = \frac{s_1}{t_1}$, and the accelerating force $p = \frac{v}{t} = \frac{s_1}{tt_1}$. If $t_1 = t = 1$, experiment gives directly $p = s_1$, and by putting the value found in the above formula, it will give the accelerating force of gravity.

Fig. 270.



§ 228. The acceleration of the motion of the weights P and Q, which are suspended to a system consisting of a fixed pulley AB, and a movable pulley EG, is given in the following manner. Let the weights of the pulleys AB and $EG = G$ and G_1, their moments of inertia Gy^2 and $G_1 y_1^{\,2}$, and the radii $CA = a$ and $DE = a_1$, therefore the masses reduced to the circumference of the wheel

Fig. 271.



$$M = \frac{G}{g} \cdot \frac{y^2}{a^2}, \text{ and } M_1 = \frac{G_1}{g} \cdot \frac{y_1^{\,2}}{a_1^{\,2}}.$$

If the weight descend a certain space s, $Q + G_1$ will then ascend by $\frac{1}{2}\,s$ (§ 151), hence the mechanical effect produced will be $Ps - (Q + G_1)\,\frac{s}{2}$; if by this descent P acquires the velocity v, then

* Attwood's Treatise on Rectilinear and Rotary Motion.

$Q + G_1$ takes the velocity $\frac{v}{2}$, and the pulley AB at its circumference the velocity v, and the pulley EG, since in rolling motion the progressive and rotary motion are equal to each other, at its circumference the velocity $\frac{v}{2}$. The sum of the *vis viva* corresponding to these masses and their velocities is $\frac{P}{g} \cdot v^2 + \frac{Q + G_1}{g} \cdot \left(\frac{v}{2}\right)^2 + \frac{Gy^2}{a^2} \cdot v^2 + \frac{G_1 y_1}{a_1^2} \cdot \left(\frac{v}{2}\right)^2$, and if their halves be equated to the mechanical effect expended, we shall obtain the equation:

$$\left(P - \frac{(Q + G_1)}{2}\right) s = \left(P + \frac{Q + G_1}{4} + \frac{Gy^2}{a^2} + \frac{G_1 y_1^2}{4 a_1^2}\right) \frac{v^2}{2g}.$$

Hence, the velocity corresponding to the space s described by P:

$$v = \sqrt{\frac{2\, gs \left(P - \frac{Q + G_1}{2}\right)}{P + \frac{Q + G_1}{4} + \frac{Gy^2}{a^2} + \frac{G_1 y_1^2}{4 a_1^2}}}$$

For the acceleration $ps = \frac{v^2}{2}$; hence, here

$$p = \left(\frac{P - \frac{Q + G_1}{2}}{P + \frac{Q + G_1}{4} + \frac{Gy^2}{a^2} + \frac{G_1 y_1^2}{4\, a_1^2}}\right) g.$$

The acceleration of $Q + G_1$ is $= \frac{p}{2}$, and the rotary acceleration of G_1 is equal to it.

The tension of the string BE, connecting both pulleys, is $S = P - \left(P + \frac{Gy^2}{a^2}\right) \frac{p}{g}$, because the force $\left(P + \frac{Gy^2}{a^2}\right) \frac{p}{g}$ is expended upon the acceleration of the motions of P and G. The tension of the fixed string GH, on the other hand:

$$S_1 = S - \frac{G_1 y_1^2}{a_1^2} \cdot \frac{p}{2g},$$

because the pulley EG is put into rotation by the difference $S - S_1$ of the tensions of the strings.

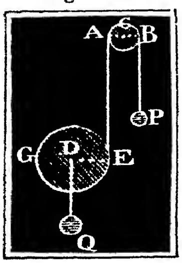
Example. In a system of pulleys, Fig. 271, the weights $P = 40$ lbs. and $Q = 66$ lbs. are suspended, and each of the solid pulleys weighs 6 lbs.; required, the acceleration of the motions of these weights. The moving force is $P - \frac{Q + G_1}{2} = 40 - \frac{66 + 6}{2}$ $= 4$ lbs., the mass of a pulley reduced to its circumference is: $\frac{Gy^2}{ga^2} = \frac{G_1 y_1^2}{g a_1^2} = \frac{G}{2g} = \frac{6}{2g} = \frac{3}{g}$ (§ 221), and the aggregate of the inert masses:

$$= \left(P + \frac{Q + G_1}{4} + \frac{Gy^2}{a^2} + \frac{G_1 y_1^2}{4\, a^2}\right) \div g = \left(40 + \frac{72}{4} + 3 + \frac{3}{4}\right) \div g = \frac{247}{4\, g},$$

hence the accelerated motion of the descending weight is: $p = \frac{4}{247} \cdot 4\ g = \frac{16 \cdot g}{247} = \frac{16 \cdot 32{,}2}{247} = \frac{515{,}2}{247} = 2{,}086$ feet; on the other hand, the accelerated motion of the ascending weight: $\frac{p}{2} = 1{,}043$ feet. The tension of the string BE is: $S = P - \left(P + \frac{G}{2}\right)\frac{p}{g} = 40 - 43 \cdot \frac{2{,}086}{32{,}2} = 40 - 2{,}782 = 37{,}218$ lbs.; that of the string GH, $= S - \frac{G}{2} \cdot \frac{p}{2g} = 37{,}218 - 3 \cdot \frac{1{,}043}{32{,}2} = 36{,}247$ lbs.

§ 229. The motion is more complicated, if the pulley EG, Fig. 272, be suspended only by a string passing round it. Let us assume that P descends with the accelerating force p, and Q ascends with the accelerating force q, we then obtain the acceleration of the rotary motion at the circumference of the loose pulley $q_1 = p - q$ (§ 42). Let us now put the tension of the strings at $AE = S$, we then obtain $P - S = \left(P + \frac{Gy^2}{a^2}\right)\frac{p}{g}$; further, $S - (Q + G_1) = (Q + G_1)\frac{q}{g}$, since from § 214, it may be assumed that S acts at the centre of gravity D of EG; and lastly, $S = \frac{G_1 y_1^2}{a_1^2} \cdot \frac{q_1}{g}$, since it may also be assumed that the centre of gravity D is fixed, and the pulley put into rotation by S. The last three formulæ give the accelerating force

Fig. 272.



$$p = \frac{P - S}{P + \frac{Gy^2}{a^2}}\,g,\quad q = \left(\frac{S - (Q + G_1)}{Q + G_1}\right)g, \text{ and } q_1 = \frac{Sa_1^2}{G_1 y_1^2}\,g, \text{ and}$$

all three being put into the equation $q_1 = p - q$, we obtain

$$\frac{Sa_1^2}{G_1 y_1^2}\,g = \frac{P - S}{P + \frac{Gy^2}{a^2}}\,g - \frac{S - (Q + G_1)}{Q + G_1}\,g,$$

whence the tension of the string follows

$$S = \frac{2\,Pa^2 + Gy^2}{\left(\frac{a_1^2}{G_1 y_1^2} + \frac{1}{Q + G_1}\right)(Pa^2 + Gy^2) + a^2}.$$

The accelerating forces are given from the value of S by the application of the above formulæ.

If we neglect the mass G of the fixed pulley, and also put $Q = 0$, we obtain simply:

$$S = \frac{2\,Pa^2 \cdot G_1 y_1^2}{P(a_1^2 + y_1^2)\,a^2 + Ga_1^2 y_1^2} = \frac{2\,PG_1 y_1^2}{G_1 y_1^2 + P(a_1^2 + y_1^2)}.$$

If the extremity of the string AE, instead of passing over the pulley AB, is fixed, we have then the accelerating force $p = 0$, hence $g_1 = -q$, and consequently the tension

$$S = \frac{(Q + G_1)\, G_1 y_1^2}{(Q + G_1)\, a_1^2 + G_1 y_1^2};\ \text{for } Q = 0:$$
$$S = \frac{G_1 y_1^2}{a_1^2 + y_1^2}.$$

If the rolling body G_1 be a solid cylinder, we have then $\frac{G_1 y_1^2}{a^2} = \frac{1}{2}\, G_1$, and for the first the tension $S = \frac{2\, PG}{3\, P + G_1}$, and for the second $S = \frac{G_1}{3}$. If in the first case the weight P descends, p is then negative, therefore:

$$S > P;\ i.\,e.\ 2\, PG_1 y_1^2 > PG_1 y_1^2 + P^2 (a_1^2 + y_1^2),$$

simply $\frac{G_1}{P} > 1 + \frac{a_1^2}{y_1^2}$; further, that G_1 may descend, it is necessary that $S < G_1$, therefore $\frac{G_1}{P_1} > 1 - \frac{a_1^2}{y_1^2}$.

Example. If when in a system of pulleys, Fig. 271, the string GH suddenly breaks, the string BE at the commencement becomes stretched by the force

$$S = \frac{2\, P + \frac{Gy^2}{a^2}}{\left(\frac{a_1^2}{G_1\, y_1^2} + \frac{1}{Q + G_1}\right)\left(P + \frac{Gy^2}{a^2}\right) + 1} = \frac{2\,.\,40 + 3}{\left(\frac{1}{3} + \frac{1}{72}\right)(40 + 3) + 1}$$
$$= \frac{83\,.\,72}{25\,.\,43 + 72} = \frac{5976}{1147} = 5{,}210 \text{ lbs.}$$

The accelerated motion of the descending weight P will be:

$$p = \left(\frac{P - S}{P + \frac{Gy^2}{a^2}}\right) g = \left(\frac{40 - 5{,}210}{40 + 3}\right).\, 32{,}2 = \frac{34{,}79}{43}\,.\, 32{,}2 = 26{,}023 \text{ feet.}$$

Further, that of the descending pulley:

$$q = \left(\frac{Q + G_1 - S}{Q + G_1}\right) g = \left(\frac{72 - 5{,}210}{72}\right) 32{,}2 = \frac{66{,}79}{72}\,.\, 32{,}2 = 30{,}0 \text{ feet;}$$

and the acceleration of rotation of this pulley:

$$q_1 = \frac{S a_1^2}{G_1\, y_1^2} \cdot g = \frac{5{,}210}{3}\,.\, 32{,}2 = 55{,}75 \text{ feet.}$$

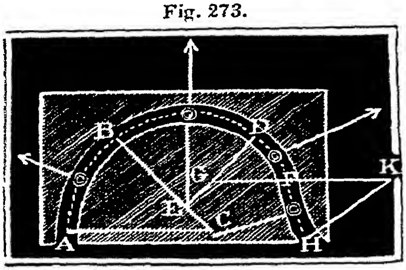
CHAPTER II.

CENTRIFUGAL FORCE.

§ 230. *Normal Force.*—When a material point moves in a curved line, it has in every point of its path an accelerated motion, deviating from that of the direction of motion, which we have become acquainted with in phoronomics under the name of the *normal accelerating force.* If the radius of curvature at any place of the path of the moving point $= r$, and its velocity $= v$, we have then for the normal accelerating force $p = \frac{v^2}{r}$ (§ 41). Let now the mass of the point $= M$, the normal

accelerating force will then be $Mp = \frac{Mv^2}{r}$, which we must regard as the first cause of the point changing its direction of motion at each position. If no other tangential force but the normal act upon the point, its velocity v will be invariable and $= c$, and hence the normal force $P = \frac{Mc^2}{r}$ will be dependent only on the curvature at each moment and on the radius of curvature, and will be greater, the greater the curvature or its radius; for double the radius of curvature, for instance, the normal force is only half as great as for a single radius of curvature. If a material point M is constrained by a horizontal path, Fig. 273, to describe a curve $ABDFH$, it will have, disregarding the friction at all places, the same velocity c, and will exert at each place a pressure equivalent to the normal force against the concave surface. During the description of the arc AB, the pressure $= \frac{Mc^2}{\overline{CA}}$, during that of $BD = \frac{Mc^2}{\overline{EB}}$, for the arc DF it $= \frac{Mc^2}{\overline{GD}}$, and for the arc $FH = \frac{Mc^2}{\overline{KF}}$, if CA, EB, GD, and KF are the radii of curvature of the portions of the path AB, BD, DF, and FH.

Fig. 273.



§ 231. *Centripetal and Centrifugal Force.*—When a material point or body moves in a circle, the normal force acts radially inwards, whence it is called the *centripetal force;* whilst the force which the body opposes, by virtue of its inertia, *i. e.*, which acts radially outwards, has received the name of *centrifugal force.* The centripetal force is that which acts directly upon the body; the centrifugal is the reacting force of the body. Each is equal in amount and opposite in direction to the other (§ 62).

In the revolution of the planets about the sun, the attractive force of the sun is the centripetal; but were the body constrained by a directrix, Fig. 273, to describe its circular orbit, this directrix would act, by its rigidity, as a centripetal force, and opposed to the centrifugal force of the body.

If, lastly, the revolving body be connected by a thread or by a rod with the centre of revolution, the elasticity of the rod will then be in equilibrium with the centrifugal force of the body, and thereby act as a centripetal force.

Let G be the weight of the revolving body, therefore its mass $M = \frac{G}{g}$, the radius of the circle in which it revolves $= r$, and the velocity of revolution $= v$; from the last § we have the centrifugal force:

$P = \frac{Mv^2}{r} = \frac{Gv^2}{gr} = 2 \cdot \frac{v^2}{2g} \cdot \frac{G}{r}$, therefore also $P : G = 2 \cdot \frac{v^2}{2g} : r$, i. e., *the centrifugal force is to the weight of the body as double the height due to the velocity is to the radius of revolution.*

If the motion be uniform, which always takes place if no other force (tangential force) than the centripetal act upon the body, the velocity $v = c$ may be expressed by the time of revolution T, if we put $c = \frac{\text{path}}{\text{time}} = \frac{2\pi r}{T}$, and hence we shall obtain for the centrifugal force:

$$P = \left(\frac{2\pi r}{T}\right)^2 \cdot \frac{M}{r} = \frac{4\pi^2}{T^2} \cdot Mr = \frac{4\pi^2}{gT^2} \cdot Gr.$$

Since $4\pi^2 = 39,4784$, and for the measure in feet $\frac{1}{g} = 0,031$, we then have more conveniently for calculation

$$P = \frac{39,4784}{T^2} \cdot Mr = 1,224 \cdot \frac{Gr}{T^2}.$$

The number n of revolutions per minute is often given, and therefore T is replaced by $\frac{60''}{n}$, whence it follows:

$$P = \frac{39,4784}{3600} n^2 Mr = 0,010966\, n^2 Mr = 0,000331\, n^2 Gr.$$

Hence, for equal times of revolution, or for an equal number of revolutions in a given time, the centrifugal force increases as the product of the mass and radius of revolution, and is inversely proportional, other circumstances being alike, to the squares of the times of revolution, or directly proportional to the squares of the number of revolutions. As $\frac{2\pi}{T}$ is the angular velocity ω, we may finally put: $P = \omega^2 \cdot Mr$.

Examples.—1. If a body of 50 lbs. weight describe a circle of 3 feet radius 400 times per minute, its centrifugal force will then be:

$$P = 0,000331 \cdot 400^2 \cdot 50 \cdot 3 = 52,96 \cdot 50 \cdot 3 = 7944 \text{ lbs.}$$

If this body be connected with an axis by a hempen cord, and the modulus of strength of the cord (§ 186) be 700 lbs., it will follow that $7944 = 7000 \cdot F$; hence the section of this cord will be : $F = \frac{7944}{7000} = 1,1342$ square inches, and its radius $D = \sqrt{\frac{4\,F}{\pi}} = \sqrt{\frac{4,5468}{3,1416}} = \sqrt{1,44} = 1,2$ inch. But for a threefold security D must be taken $= 1,2\sqrt{3} = 1,2 \cdot 1,732 = 2,077$ inches.—2. From the earth's radius $r = 20\frac{1}{4}$ millions of feet, and the time of rotation or length of a day $T = 24$ h. $= 24 \cdot 60 \cdot 60 = 86400''$, the centrifugal force of a body at the earth's equator is $P = 1,224 \cdot \frac{G \cdot 20,250000}{86400^2} = \frac{2478}{864^2} \cdot G = \frac{1}{300} \cdot G$ nearly; but were the length of the day $\frac{1}{17}$th part only, then: $\frac{24}{17} = 1$ h. 24 m. 42 sec. this force would be $17^2 = 289$ times as great, therefore equal to about the weight of the body. At the equator, therefore, the centrifugal force would be equivalent to that of gravity, and the body would neither rise nor fall. In the revolution of the moon about the earth, its centrifugal force is counteracted by the attraction of the earth. Let G be the weight of the moon, r its distance from the earth, and T its time of revolution; the centrifugal force of this

heavenly body $= 1,224 \,.\, \frac{Gr}{T^2}$. Let a be the radius of the earth, and let us assume that the force of gravity at different distances from its centre increases inversely as a power of these distances; we have then the gravity of the moon or the attractive force of the earth $= G\left(\frac{a}{r}\right)^n$, and if we equate both forces to each other, we then obtain $\left(\frac{a}{r}\right)^n = 1,224 \,.\, \frac{r}{T^2}$.

Now $\frac{a}{r} = \frac{1}{60}$, $r = 1250$ million feet, and $T = 27$ days, 7 hours, 42 minutes $= 39342$ minutes $= 39342 \,.\, 60$ seconds; hence it follows:

$$\left(\frac{1}{60}\right)^n = \frac{1,229 \,.\, 1250}{393,4^2 \,.\, 36} = \frac{1}{3600}, \text{ nearly } = \left(\frac{1}{60}\right)^2$$

and hence, $n = 2$; *i. e.*, the gravitating force of the earth is in an inverse ratio to the square of the distance.

§ 232. *Centrifugal Forces of Extended Masses.*—For any system of masses, or for a mass of finite extension, the formula above found for the centrifugal force is not directly applicable, because we know not beforehand what radius of gyration we have to introduce into the calculation. To find this, we proceed in the following manner. In Fig. 274, let CZ be the axis of revolution, CX and CY its two rectangular co-ordinate axes, further let M be a particle M_1, and $MK = x$, $MM = y$, and $MN = z$, its distances from the co-ordinate planes YZ, XZ and XY. As the centrifugal force P acts radially, its point of application may be transferred to its point of intersection O with the axis of revolution. If now we resolve this force in the direction of the axes CX and CY, we shall obtain the component forces $OQ = Q$ and $OR = R$, for which $OQ : OP = OL : OM$, and $OR : OP = OK : OM$, whence $Q = \frac{x}{r} P$ and $R = \frac{y}{r} P$, where r represents the distance OM of the particle from the axis of revolution. Let us proceed in a similar manner with all the particles, and we shall obtain two systems of parallel forces, one in the plane XZ, and the other in the plane YZ, but each acting perpendicularly to the axis CZ. For distinction, let us avail ourselves of the index numbers 1, 2, 3, &c., and, therefore, put for the particles of the mass, M_1, M_2, M_3, and for their distances x_1, x_2, x_3, &c., we shall obtain the resultant of the one system, Fig. 275,

Fig. 274.

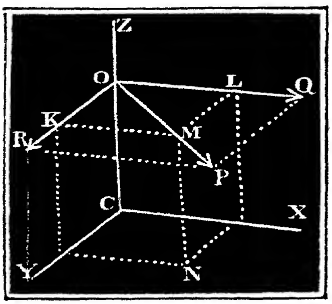
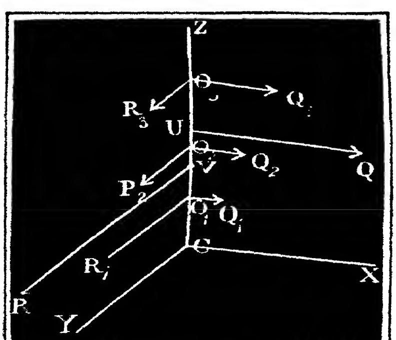


Fig. 275.



$$Q = Q_1 + Q_2 + Q_3 + \ldots$$
$$= \frac{P_1 x_1}{r_1} + \frac{P_2 x_2}{r_2} + \frac{P_3 x_3}{r_3} + \ldots =$$

$\omega^2 . (M_1 x_1 + M_2 x_2 + \ldots)$ and that of the other $R = R_1 + R_2 + \ldots = \omega^2 . (M_1 y_1 + M_2 y_2 + \ldots)$. Let us finally put the distances of the particles from the plane XY, CO_1, CO_2, &c., $= z_1, z_2$, &c., we shall obtain for the points of application of these resultants the distances $CU = u$, and $CV = v$ by the equations $(Q_1 + Q_2 + \ldots) u = Q_1 z_1 + Q_2 z_2 + \ldots$ and $(R_1 + R_2 + \ldots) v = R_1 z_1 + R_2 z_2 + \ldots$, whence it follows:

$$u = \frac{Q_1 z_1 + Q_2 z_2 + \ldots}{Q_1 + Q_2 + \ldots} = \frac{M_1 x_1 z_1 + M_2 x_2 z_2 + \ldots}{M_1 x_1 + M_2 x_2 + \ldots}, \text{ and}$$

$$v = \frac{R_1 z_1 + R_2 z_2 + \ldots}{R_1 + R_2 + \ldots} = \frac{M_1 y_1 z_1 + M_2 y_2 z_2 + \ldots}{M_1 y_1 + M_2 y_2 + \ldots}.$$

Hence, therefore, in general, the centrifugal forces of a system of bodies, or an extended body, may be reduced to two forces, which, so long as u and v are unequal, cannot be resolved into a single one.

Example. The masses of a system are

	$M_1 = 10$ lbs.,	$M_2 = 15$ lbs.,	$M_3 = 18$ lbs.,	$M_4 = 12$ lbs.,
and their distances	$x_1 = 0$ inches,	$x_2 = 4$ inches,	$x_3 = 2$ inches,	$x_4 = 6$ inches,
	$y_1 = 3$ "	$y_2 = 1$ "	$y_3 = 5$ "	$y_4 = 3$ "
	$z_1 = 2$ "	$z_2 = 3$ "	$z_3 = 3$ "	$z_4 = 0$ "

we have then the following mean centrifugal forces
$Q = \omega^2 . (10 . 0 + 15 . 4 + 18 . 2 + 12 . 6) = 168 . \omega^2$, and
$R = \omega^2 . (10 . 3 + 15 . 1 + 18 . 5 + 12 . 3) = 171 . \omega^2$, and the distances of their points of application from the origin C:

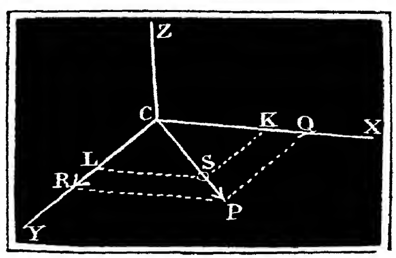
$$u = \frac{10 . 0 . 2 + 15 . 4 . 3 + 18 . 2 . 3 + 12 . 6 . 0}{10 . 0 + 15 . 4 + 18 . 2 + 12 . 6} = \frac{288}{168} = \frac{12}{7} = 1{,}714 \text{ inches and}$$

$$v = \frac{10 . 3 . 2 + 15 . 1 . 3 + 18 . 5 . 3 + 12 . 3 . 0}{10 . 3 + 15 . 1 + 18 . 5 + 12 . 3} = \frac{375}{171} = \frac{125}{57} = 2{,}193 \text{ inches.}$$

The difference of these two values of u and v shows that the centrifugal forces cannot be replaced by a single force.

§ 233. If the particles of the mass lie in a plane at right angles to the axis of revolution, Fig. 276, their centrifugal forces may be reduced to a single force, because their directions intersect at a point in the axis. Retaining the denominations of the former §, we shall obtain the resultant centrifugal force in this case:

Fig. 276.



$P = \sqrt{Q^2 + R^2} = \omega^2 \sqrt{[(M_1 x_1 + M_2 x_2 + \ldots)^2 + (M_1 y_1 + M_2 y_2 + \ldots)^2]}$. Now $CK = x$, and $CL = y$, are the co-ordinates of the centre of gravity of the system $M = M_1 + M_2 + \ldots$, we then have:

$$M_1 x_1 + M_2 x_2 + \ldots = Mx \text{ and } M_1 y_1 + M_2 y_2 + \ldots = My,$$

hence the centrifugal force:

$$P = \omega^2 \sqrt{M^2 x^2 + M^2 y^2} = \omega^2 M \sqrt{x^2 + y^2} = \omega^2 Mr,$$

provided further that $r = \sqrt{x^2 + y^2}$ represent the distance CS of the centre of gravity from the axis of revolution CZ.

For the angle $PCX = \alpha$, which this force includes with the axis

CX, *tang.* $\alpha = \frac{R}{Q} = \frac{My}{Mx} = \frac{y}{z}$; *hence the direction of the centrifugal force passes through the centre of gravity of the system, and centrifugal force is exactly the same as if the collective masses were united at the centre of gravity.*

Fig. 277.

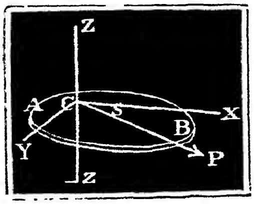
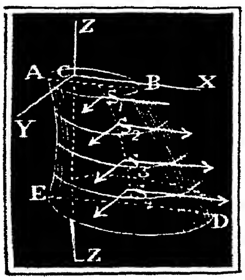


Fig. 278.



For a disc AB at right angles to the axis of revolution $\overline{ZZ}$, Fig. 277, the centrifugal force is from this $= \omega^2 Mr$, if M represents its mass, and r the distance CS of its centre of gravity S from the axis. To find the centrifugal force of another body $ABDE$, Fig. 278, we must decompose it into elementary discs by planes at right angles to the axis $\overline{ZZ}$, find their centres of gravity S_1, S_2, &c., and determine the centrifugal forces by help of these last, decompose each of them in the direction of the axes CX and CY into component forces, and reduce the forces in the plane ZCX to a resultant Q, and those in the plane ZCY to a resultant R.

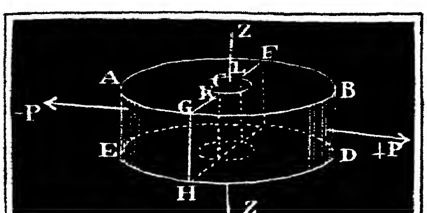
If the centres of gravity of the aggregate discs lie in a line parallel to the axis of revolution, $x = x_1 = x_2$, &c., and $y = y_1 = y_2$, &c., and therefore also $r = r_1 = r_2$, &c., hence the centrifugal force of the whole body $P = \omega^2 (M_1 r + M_2 r + \ldots) = \omega^2 Mr$, and the distance of its point of application from the plane XY:

$$z = \frac{(M_1 z_1 + M_2 z_2 + \ldots) r}{(M_1 + M_2 + \ldots) r} = \frac{M_1 z_1 + M_2 z_2 + \ldots}{M_1 + M_2 + \ldots}.$$

According to these equations, the centrifugal force of a body, whose elements are in a line parallel to the axis, is equivalent to the centrifugal force of the mass of this body, reduced to its centre of gravity, and its point of application and centre of gravity coincide. From this the centrifugal forces of all rotary bodies, whose geometric axes run parallel with the axis of rotation may be found. If the geometric axis of any such body coincide with the axis of rotation, the centrifugal force is equal to nothing.

Example.—The dimensions, the density, and strength of a millstone, $ABDE$, Fig. 279, are given; it is required to find the angular velocity ω, in consequence of which rupture will take place in virtue of centrifugal force. If we put the radius of the millstone $= r_1$, the radius CK of its eye $= r_2$, the height $AE = GH = l$, the density $= \gamma$, and the modulus of strength $= K$, we obtain the force required for rupture $= 2 (r_1 - r_2) lK$, the weight of the stone $G = \pi (r_1^2 - r_2^2) l\gamma$, and the radius of gyration of each half of the stone, *i. e.* the distance of its centre of gravity from the axis of rotation (§ 109), $r = \frac{4}{3\pi} \cdot \frac{r_1^3 - r_2^3}{r_1^2 - r_1^2}$. At the

Fig. 279.



moment of rupture, the centrifugal force of half the stone is equivalent to the strength; we hence obtain the equation of condition $\omega^2 \cdot \frac{1}{2} \frac{Gr}{g} = 2 (r_1 - r_2)\, lK$, i. e. $\omega^2 \cdot \frac{2}{3} (r_1^3 - r_2^3) \frac{l\gamma}{g} = 2 (r_1 - r_2)\, lK$, or leaving out $2\, l$ on both sides, it follows that

$$\omega = \sqrt{\frac{3\, g\, (r_1 - r_2)\, K}{(r_1^3 - r_2^3)\, \gamma}} = \sqrt{\frac{3\, g\, K}{(r_1^2 + r_1 r_2 + r_2^2)\, \gamma}}.$$

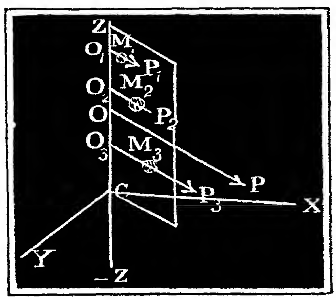
If $r_1 = 2$ feet $= 24$ inches, $r_2 = 4$ inches, $K = 750$ lbs., and the specific gravity of the millstone $= 2{,}5$, therefore the weight of a cubic inch of its mass $= \frac{62{,}5 \times 2{,}5}{1728} = 0{,}0903$ lbs., it follows that the angular velocity at the moment of rupture is:

$$\omega = \sqrt{\frac{3 \cdot 12 \cdot 32{,}2 \cdot 750}{688 \cdot 0{,}0903}} = \sqrt{\frac{869400}{62{,}1264}} = 112{,}1 \text{ inches.}$$

If the number of rotations per minute $= n$, we have then $\omega = \frac{2\,\pi\, n}{60}$; hence, inversely, $n = \frac{30\,\omega}{\pi}$, but here $= \frac{30 \cdot 112{,}1}{\pi} = 1070$. The average number of rotations of such a millstone is only 120, therefore 9 times less.

§ 233. If the collective particles M_1, M_2, of a system of masses, Fig. 280, or the centres of gravity of the elements of a body lie in a plane passing through their axis of revolution, the centrifugal forces will then form a system of parallel forces, and these may be reduced according to the rule to a single force. The distances of the particles or the elements from the axis of revolution $\overline{ZZ}$, are $O_1 M_1 = r_1$, $O_2 M_2 = r_2$, &c., we obtain for their centrifugal forces:

Fig. 280.



$$P_1 = \omega^2 M_1 r_1,\ P_2 = \omega^2 M_2 r_2, \text{ \&c.,}$$

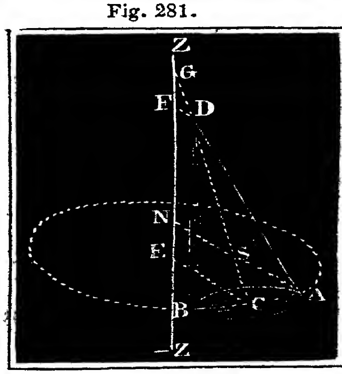
and hence the resultant centrifugal force:

$$P = \omega^2 (M_1 r_1 + M_2 r_2 + \dots) = \omega^2 M r,$$

r representing the distance of the centre of gravity of the mass M from the axis of revolution. Therefore, here also the distance of the centre of gravity from the axis of revolution must be regarded as the radius of gyration. But to find the point of application O of the resultant centrifugal force, let us put the distances of the particles of the mass from the normal plane: $CO_1 = z_1$, $CO_2 = z_2$, &c., into the formula:

$$CO = z = \frac{M_1 r_1 z_1 + M_2 r_2 z_2 + \dots}{M_1 r_1 + M_2 r_2 + \dots}.$$

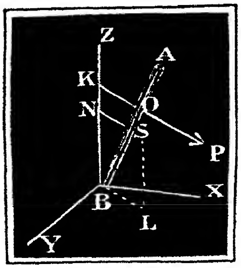
Fig. 281.



By help of the formula $P = \omega^2 Mr$, the centrifugal forces of rotary bodies and other geometric bodies may be found, when their axes and the axis of revolution lie in one plane. The centrifugal force of a right cone ADB, Fig. 281, may be found, if the distance SN of its centre of gravity S from the axis of revolution $\overline{ZZ}$ be put as r into the formula. If the height of the cone $CD = h$, the distance DF of the point D from the axis

of revolution $= a$, and the angle CGE, by which the geometrical axis deviates from the axis of revolution $\overline{ZZ} = \alpha$, we have then $r = a + \frac{3}{4} h$ $sin.\ \alpha$. For a rod AB, Fig. 282, whose length $AB = l$, and angle of inclination ABZ to the axis of revolution $BZ = \alpha$, we have $r = SN = \frac{1}{2}\ l\ sin.\ \alpha$, therefore, the centrifugal force:

Fig. 282.



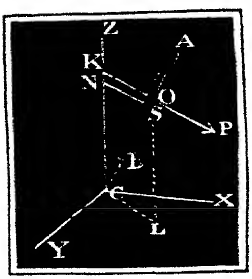
$P = \omega^2 . \frac{1}{2}\ M\ l\ sin.\ \alpha$; but to find the point of application O of this force, in the expression $\omega^2 . \frac{M}{n}\ x\ sin.\ \alpha\ .\ x\ cos.\ \alpha = \omega^2 . \frac{M}{n}\ x^2\ sin.\ \alpha\ cos.\ \alpha$ for the moment of the element $\frac{M}{n}$ of the rod, let us put for x successively the values $\frac{l}{n}, \frac{2\,l}{n}, \frac{3\,l}{n}$, &c., and add the results, in this manner we shall obtain the moment of the entire rod:

$$Pz = \omega^2 \frac{M}{n}\ sin.\ \alpha\ cos.\ \alpha\ \frac{l^2}{n^2}\ (1^2 + 2^2 + 3^2 + \ldots + n^2)$$

$$= \tfrac{1}{3}\ \omega^2\ Ml^2\ sin.\ \alpha\ cos.\ \alpha,$$

$z = \frac{1}{3}\ \omega^2\ Ml^2\ sin.\ \alpha\ cos.\ \alpha : \frac{1}{2}\ \omega^2\ Ml\ sin.\ \alpha = \frac{2}{3}\ l\ cos.\ \alpha$, and the distance of the point of application O from the extremity of the rod B lying in the axis, $BO = \frac{2}{3}\ l$.

Fig. 283.



If the rod AB, Fig. 283, does not reach the axis, we then have:

$P = \frac{1}{2}\ \omega^2\ F\,l_1^{\,2}\ sin.\ \alpha - \frac{1}{2}\ \omega^2\ F\,l_2^{\,2}\ sin.\ \alpha = \frac{1}{2}\ \omega^2\ F$ $sin.\ \alpha\ (l_1^{\,2} - l_2^{\,2})$, and the moment:

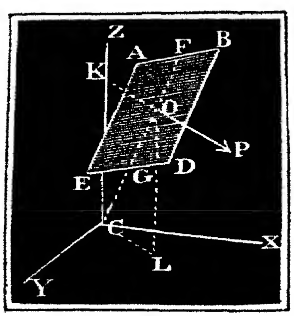
$$Pz = \tfrac{1}{3}\ \omega^2\ F\ sin.\ \alpha\ cos.\ \alpha\ (l_1^{\,3} - l_2^{\,3}),$$

because the mass of CA, $=$ the cross section into the length, $= Fl_1$, and the mass of $CB = Fl_2$, hence it follows that the distance of the point of application O from its intersection with the axis C is:

$$CO = \tfrac{2}{3}\ \frac{l_1^{\,3} - l_2^{\,3}}{l_1^{\,2} - l_2^{\,2}} = l + \frac{(l_1 - l_2)^2}{12\,l},$$

where l expresses the distance CS of the centre of gravity, but $l_1 - l_2$ the length of the rod AB.

Fig. 284.



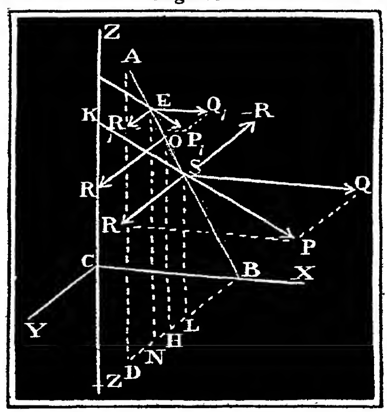
This formula is also applicable to a rectangular plate $ABDE$, Fig. 284, which is divided into two congruent right angles by the plane of the axis COZ, because the centrifugal force acts at the middle of each of the elements, which are obtained by sections normal to CZ. Therefore the distances CF and CG of the two bases AB and DE from the point C of the axis, are l_1 and l_2, we have here also $CO = \frac{2}{3} \cdot \frac{l_1^{\,3} - l_2^{\,3}}{l_1^{\,2} - l_2^{\,2}}$.

§ 234. In the case where the particles of the body neither lie in a plane normal to, nor in a plane passing through the axis of revolution, the resultant centrifugal forces,
$Q = \omega^2 (M_1 x_1 + M_2 x_2 + \ldots)$ and $R = \omega^2 (M_1 y_1 + M_2 y_2 + \ldots)$ cannot be reduced to a single force, nevertheless it is possible to replace these forces by a force acting at the centre of gravity:

$$P = \sqrt{Q^2 + R^2} = \omega^2 Mr,$$

and by a couple composed of Q and R. If, namely, we apply to the centre of gravity S, four forces $+ Q$ and $- Q$, $+ R$ and $- R$ balancing each other, the positive parts will give a resultant $P = \omega^2 \sqrt{Q^2 + R^2}$, and the negative parts on the other hand, $-Q$ and $-R$, will form the couples $(Q, -Q)$ and $(R, -R)$ with the centrifugal forces applied at U and V, which may be reduced to a single couple. In order to make ourselves acquainted with this reduction of the centrifugal forces of a rotary body, let us take the following simple case. Let the bar AB, Fig. 285, which turns about the axis $\overline{ZZ}$, lie parallel to the plane YZ, and rest with its extremity B on the axis CX. Let the length of this bar $= l$, its weight $= G$, the angle BAD at which it is inclined to the axis of rotation $= \alpha$, and its distance CB from the plane YZ, which is also its shortest distance from the axis $\overline{ZZ} = a$. Let now E be an element $\frac{M}{n}$ of the bar, and $BE = x$ its distance from the extremity B, we shall then have the projection $BN = x \sin. \alpha$, and hence the components of the centrifugal force P_1 of this element:

Fig. 285.



$Q_1 = \omega^2 . \frac{M}{n} . CB = \omega^2 . \frac{M}{n} a$ and $R_1 = \omega^2 . \frac{M}{n} . BN = \omega^2 . \frac{M}{n} x \sin. \alpha$, and their moments about the principal plane XCY:

$Q_1 z_1 = \omega^2 . \frac{M}{n} . CB . EN = \omega^2 . \frac{M}{n} ax \cos. \alpha$ and $R_1 z_1 = \omega^2 . \frac{M}{n} x^2 \sin. \alpha \cos. \alpha$. The several components parallel to the plane XZ give the resultant $Q = Q_1 + Q_2 + \ldots = n . \omega^2 . \frac{M}{n} a = \omega^2 . Ma$ and its moment $Qu = Q_1 z_1 + Q_2 z_2 + \ldots = \omega^2 . \frac{M}{n} a \cos. \alpha (x_1 + x_2 + \ldots)$;

or, as x_1 is to be taken $= \frac{l}{n}$, $x_2 = 2 \frac{l}{n}$, $x_3 = \frac{3l}{n}$, &c.,

$$Qu = \omega^2 . \frac{M}{n} a \cos. \alpha . \frac{l}{n} (1 + 2 + 3 + \ldots + n) = \omega^2 . \frac{M}{n} a \cos. \alpha .$$

$\frac{l}{n} \cdot \frac{n^2}{2} = \frac{1}{2} \omega^2 \cdot Mal \; cos. \; \alpha$; the distance, therefore, of the point of application of this component from the plane XY is:

$$LS = u = \frac{\frac{1}{2} \omega^2 \; Mal \; cos. \; \alpha}{\omega^2 \; Ma} = \tfrac{1}{2} \; l \; cos. \; \alpha,$$

i. e. it coincides with the centre of gravity of the bar. The components, which act parallel to YZ, give the resultant $R = R_1 + R_2 + \ldots = \omega^2 \cdot \frac{M}{n} \; sin. \; \alpha \; (x_1 + x_2 + \ldots) = \omega^2 \cdot \frac{M}{n} \; sin. \; \alpha \; \frac{l}{n} \cdot \frac{n^2}{2} = \frac{1}{2} \omega^2 \; Ml \; sin. \; \alpha$ with the moment $\omega^2 \cdot \frac{M}{n} sin. \; \alpha \; cos. \; \alpha \; (x_1^2 + x_2^2 + \ldots) = \omega^2 \cdot \frac{M}{n} \; sin. \; \alpha \; cos. \; \alpha \cdot \left(\frac{l^2}{n^2} + \frac{4l^2}{n^2} + \ldots\right) = \omega^2 \cdot \frac{M}{n} \frac{l^2}{n^2} \; sin. \; \alpha \; cos. \; \alpha \; (1 + 4 + 9 + \ldots + n^2) = \omega^2 \cdot \frac{M}{n} \cdot \frac{l^2}{n^2} sin. \; \alpha \; cos. \; \alpha \cdot \frac{n^3}{3} = \frac{1}{3} \; \omega^2 \; M \; l^2 \; sin. \; \alpha \; cos. \; \alpha$; the distance of the point of application of the force from the plane XY is : $HO = v = \frac{\frac{1}{3} \omega^2 \; M \; l^2 \; sin. \; \alpha \; cos. \; \alpha}{\frac{1}{2} \; \omega^2 \; M \; l \; sin. \; \alpha} = \frac{2}{3} \; l \; cos. \; \alpha$, *i. e.* this point lies about $(\frac{2}{3} - \frac{1}{2}) \; l \; cos. \; \alpha = \frac{1}{6} \; l \; cos. \; \alpha = \frac{1}{6}$ of the projection AD parallel to the axis above the centre of gravity S of the bar.

From the forces $Q = \omega^2 \; M \; a$, and $R = \frac{1}{2} \; \omega^2 \; M \; l \; sin. \; \alpha$, the final resultant applied at the centre of gravity of the bar follows: $P = \sqrt{Q^2 + R^2} = \omega^2 \; M \sqrt{a^2 + \frac{1}{4} \; l^2 \; sin. \; \alpha^2}$, and the couple $(R_1 - R)$ with the moment

$$R \cdot \overline{SO} = \tfrac{1}{2} \; \omega^2 \; M \; l \; sin. \; \alpha \cdot \tfrac{1}{6} \; l = \tfrac{1}{12} \; \omega^2 \; M \; l^2 \; sin. \; \alpha.$$

§ 235. *Free Axes.*—In general, the centrifugal forces of a body revolving uniformly about an axis, exert a pressure upon the axis; it is, nevertheless, possible that these forces mutually counteract each other, and for this reason the axis will have no pressure to sustain. This case presents itself, for instance, in every solid of rotation revolving about its geometric axis, or its axis of symmetry, and especially in the wheel and axle, and in the water-wheel, &c. If, under these circumstances, no external forces act upon a rotary body, or upon such a system, the body will remain for ever in this state of revolution, without its being necessary that the axis of revolution should be fixed. This axis is called, for this reason, a *free axis.* From the preceding, the conditions immediately follow by which an axis of revolution becomes a *free axis.* It is requisite not only that the resultants P and Q of the components of the centrifugal forces acting parallel to the planes of the axes XZ and YZ, but that the sum of the statical moments of each of the two systems of forces, should $= 0$. Hence, from this:

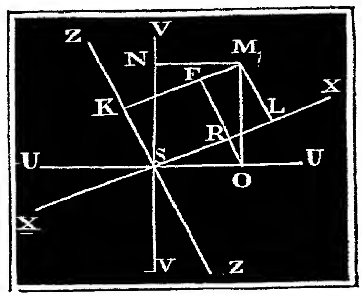
1. $M_1 x_1 + M_2 x_2 + \ldots = 0,$
2. $M_1 y_1 + M_2 y_2 + \ldots = 0,$ further
3. $M_1 x_1 z_1 + M_2 x_2 z_2 + \ldots = 0,$ and
4. $M_1 y_1 z_1 + M_2 y_2 z_2 + \ldots = 0.$

The two first equations require that the free axis pass through the centre of gravity of the body or system. The two last afford the elements for determining the position of this axis. It may, besides, be proved that every body or system has at least three *free axes*, and that these axes meet at right angles in the centre of gravity of the system.

The higher mechanics distinguish other axes besides the free, which run parallel to these and intersect each other in a point of the system, and are called the *principal axes*. It may be also proved that the moment of inertia of a body about one of the principal axes is a maximum, about a second axis a minimum, and about a third neither one nor the other.

§ 236. If the particles of a mass lie in one plane, for instance, if the mass forms a thin plate or plane figure, then the straight line passing through the centre of gravity of the entire mass, and normal to its plane, is a free axis of the mass; for in this case the mass has no radius of gyration, and hence the only possible centrifugal force is $= 0$. To find the other two free axes, let us proceed in the following manner. Let S, Fig. 286, be the centre of gravity of a mass, and let $U\overline{U}$ and $V\overline{V}$ be two co-ordinate axes in the plane of the mass; let us determine the molecules by co-ordinates parallel to these axes, viz. the molecules M by the co-ordinates $M_1N = u_1$ and $M_1O = v_1$. Let $X\overline{X}$, on the other hand, be a free axis, $Z\overline{Z}$ an axis perpendicular to it; further, let the angle to be determined XSU, which the free axis makes with the co-ordinate axis SU, $= \phi$, and let the co-ordinates of the particles referred to the axes $X\overline{X}$ and $Z\overline{Z}$: be $x_1, x_2 \ldots, z_1, z_2 \ldots$, therefore for the particle $M_1 : M_1K = x_1$ and $M_1L = z_1$. From this we easily obtain:

Fig. 286.



$x_1 = M_1K = SR + RL = SO\ cos.\ \phi + OM_1\ sin.\ \phi = u_1\ cos.\ \phi + v_1\ sin.\ \phi$
$z_1 = M_1L = -OR + OF = -SO\ sin.\ \phi + OM_1\ cos.\ \phi = -u_1\ sin.\ \phi + v_1\ cos.\ \phi$; and hence the product:

$$x_1z_1 = (u_1\ cos.\ \phi + v_1\ sin.\ \phi)\ (-u_1\ sin.\ \phi + v_1\ cos.\ \phi)$$
$$= -(u_1^2 - v_1^2)\ sin.\ \phi\ cos.\ \phi + u_1v_1\ (cos.\ \phi^2 - sin.\ \phi^2)$$

or, since $sin.\ \phi\ cos.\ \phi = \frac{1}{2}\ sin.\ 2\ \phi$ and $cos.\ \phi^2 - sin.\ \phi^2 = cos.\ 2\ \phi$, $x_1z_1 = -\frac{1}{2}(u_1^2 - v_1^2)\ sin.\ 2\ \phi + u_1v_1\ cos.\ 2\ \phi$, and hence the moment of the particle M_1:

$M_1x_1z_1 = -\frac{M_1}{2}(u_1^2 - v_1^2)\ sin.\ 2\ \phi + M_1u_1v_1\ cos.\ 2\ \phi$. The moment of the particle M_2:

$M_2x_2z_2 = -\frac{M_2}{2}(u_2^2 - v_2^2)\ sin.\ 2\ \phi + M_2u_2v_2\ cos.\ 2\ \phi$, &c., and

the sum of the moments of all the particles, or the moment of the entire mass:

$$M_1x_1z_1 + M_2x_2z_2 + \ldots = -\tfrac{1}{2}\,sin.\,2\,\phi\,[\,(M_1u_1^2 + M_2u_2^2 + \ldots) - (M_1v_1^2 + M_2v_2^2 + \ldots)\,] + cos.\,2\,\phi\,(M_1u_1v_1 + M_2u_2v_2 + \ldots).$$

That $\overline{XX}$ may become a free axis, its moment from the former paragraph must be $= 0$; hence we must put

$$\tfrac{1}{2}\,sin.\,2\,\phi\,[\,(M_1u_1^2 + M_2u_2^2 + \ldots) - (M_1v_1^2 + M_2v_2^2 + \ldots)\,] - cos.\,2\,\phi\,(M_1u_1v_1 + M_2u_2v_2 + \ldots) = 0,$$

and from this we obtain the equation of condition:

$$tang.\,2\,\phi = \frac{sin.\,2\,\phi}{cos.\,2\,\phi} = \frac{2\,(M_1u_1v_1 + M_2u_2v_2 + \ldots)}{(M_1u_1^2+M_2u_2^2+\ldots)-(M_1v_1^2+M_2v_2^2+\ldots)}$$
$$= \frac{\text{twice the moment of the centrifugal force}}{\text{difference of the moments of inertia}}.$$

By this formula, two values for $2\,\phi$ are given, which vary 180° from each other, and therefore also two values of ϕ, which vary 90° from each other; on this account, not only is the axis $\overline{XX}$ determined by this angle ϕ, a free axis, but also the axis $\overline{ZZ}$ perpendicular to it.

§ 237. The free axes of many surfaces and bodies are known without any calculation. In symmetrical figures, for instance, the axis of symmetry is a free axis, the perpendicular to the centre of gravity is a second, and the axis perpendicular to the plane of the figure a third free axis. The axis of rotation $\overline{ZZ}$ of a rotary body AB, Fig. 287, is a free axis, so is every normal $\overline{XX}$, $\overline{YY}$ . . to this, passing through the centre of gravity S. Every diameter of a sphere is a free axis, the axes $\overline{XX}$, $\overline{YY}$, $\overline{ZZ}$, of a right parallelopiped ABD,

Fig. 287.

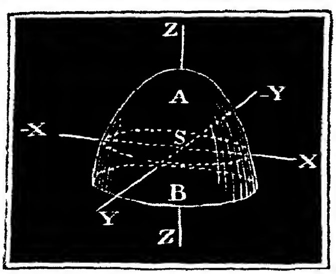


Fig. 288.

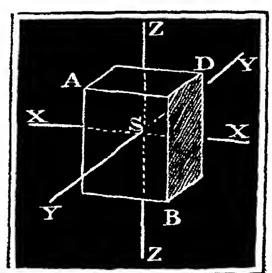
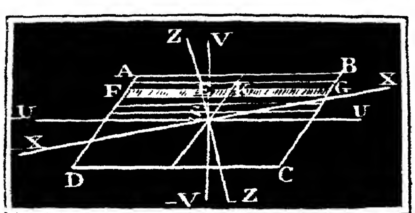


Fig. 288, bounded by six rectangles, passing through the centre of gravity S, and normal to the sides BD, AB, and AD, or running parallel with the edges, are free axes.

Fig. 289.



Let us now determine the free axes of an acute angled parallelogram $ABCD$, Fig. 289. Let us draw through its centre of gravity S, the co-ordinate axes $\overline{UU}$ and $\overline{VV}$ at right angles to each other, so that one of the sides AB of the

parallelogram may run parallel to it, and let us decompose the parallelogram by parallel lines into $2n$ equal strips, such as FG. If, now, one side $AB = 2a$, the other $AD = 2b$, and the angle ADC between the two sides $= \alpha$, we then obtain for the strip FG, distant from $\overline{UU}$, $SE = x$, the length of one part:

$$EG = KG + EK = a + x \, cotg. \, \alpha,$$

and that of the other $EF = a - x \, cotang. \, \alpha$, and since $\frac{b}{n} \, sin. \, \alpha$ is the breadth of both, the area of these strips $= \frac{b \, sin. \, \alpha}{n} (a + x \, cotg. \, \alpha)$ and $\frac{b \, sin. \, \alpha}{n} (a - x \, cotg. \, \alpha)$; the measure of the centrifugal forces about the axis $\overline{VV}$ is therefore:

$$= \frac{b \, sin. \, \alpha}{n} (a + x \, cotg. \, \alpha) \, . \, \tfrac{1}{2} (a + x \, cotg. \, \alpha) = \frac{b \, sin. \, \alpha}{2n} (a + x \, cotg. \, \alpha)^2$$

and $\frac{b \, sin. \, \alpha}{2n} (a - x \, cotg. \, \alpha)^2$, and their moments about the axis $\overline{UU}$: $\frac{b \, sin. \, \alpha}{2n} (a + x \, cotg. \, \alpha)^2 x$, and $\frac{b \, sin. \, \alpha}{2n} (a - x \, cos. \, \alpha)^2 x$. As both the forces about $\overline{VV}$ act opposite to each other, the uniting of their moments gives the difference:

$$\frac{b \, x \, sin. \, \alpha}{2n} [(a + x \, cotg. \, \alpha)^2 - (a - x \, cotg.)^2] = \frac{2}{n} abx^2 \, cos. \, \alpha.$$

If we substitute in this formula for x the values:

$$\frac{b \, sin. \, \alpha}{n}, \frac{2 \, b \, sin. \, \alpha}{n}, \frac{3 \, b \, sin. \, \alpha}{n}, \text{ \&c.,}$$

successively, and add the results, we shall obtain the measure of the moment of the centrifugal force of half the parallelogram:

$$\frac{2 \, ab}{n} cos. \, \alpha \, . \, \frac{b^2 sin. \, \alpha^2}{n^2} (1^2 + 2^2 + 3^2 + \, . \, . + n^2) = 2 \, ab^3 \, sin. \, \alpha^2 \, cos. \, \alpha \, . \, \frac{n^3}{3n^3}$$

$= \frac{2}{3} ab^3 \, sin. \, \alpha^2 \, cos. \, \alpha$, and, therefore, for the whole parallelogram, or

$M_1 u_1 v_1 + M_2 u_2 v_2 + \, . \, . \, . = \frac{4}{3} ab^3 \, sin. \, \alpha^2 \, cos. \, \alpha$. The moment of inertia of a strip FG about the axis $\overline{VV}$ is:

$$= \frac{b \, sin. \, \alpha}{n} \left(\frac{(a + x \, cotg. \, \alpha)^3}{3} + \frac{(a - x \, cotg. \, \alpha)^3}{3} \right)$$

$= \frac{2 \, b \, sin. \, \alpha}{3n} (a^3 + 3 \, ax^2 \, cotg. \, \alpha^2) = \frac{2}{3} \frac{ab}{n} sin. \, \alpha \, (a^2 + 3 \, x^2 \, cotg. \, \alpha^2)$; if now we substitute in succession for x:

$$\frac{b \, sin. \, \alpha}{n}, \frac{2 \, b \, sin. \, \alpha}{n}, \frac{3 \, b \, sin. \, \alpha}{n}, \text{ \&c.,}$$

and sum the resulting values, we shall have the moment of inertia of a half $= \frac{2}{3} ab \, sin. \, \alpha \, (a^2 + b^2 \, cos. \, \alpha^2)$, and hence that of the whole

$= \frac{4}{3}\, ab\, sin.\, \alpha\, (a^2+b^2\, cos.\, \alpha^2)$. On the other hand, the moment of inertia of the parallelogram about the axis of revolution $\overline{UU}$ is $= 4\, ab\, sin.\, \alpha\, .\, \frac{b^2\, sin.\, \alpha^2}{3} = \frac{4}{3}\, ab^3\, sin.\, \alpha^3$ (§ 220); hence the difference of the moments of inertia sought, *i. e.*

$$(M_1 u_1^2 + M_2 u_2^2 + \ldots) - M_1 v_1^2 + M_2 v_2^2 + \ldots),$$

$$= \frac{4}{3}\, ab\, sin.\, \alpha\, (a^2+b^2\, cos.\, \alpha^2) - \frac{4}{3}\, ab^3\, sin.\, \alpha^3$$

$$= \frac{4}{3}\, ab\, sin.\, \alpha\, [a^2+b^2\, (cos.\, \alpha^2 - sin.\, \alpha^2)]$$

$$= \frac{4}{3}\, ab\, sin.\, \alpha\, (a^2+b^2\, cos.\, 2\, \alpha).$$

Lastly for the angle $USX = \phi$, which the *free axis* $\overline{XX}$ makes with the co-ordinate axis $\overline{UU}$ or the side AB from § 236:

$$tang.\, 2\, \phi = \frac{2\, (M_1 u_1 v_1 + M_2 u_2 v_2 + \ldots)}{(M_1 u_1^2 + M_2 u_2^2 + \ldots) - (M_1 v_1^2 + M_2 v_2^2 + \ldots)}$$

$$= \frac{2 \frac{4}{3}\, ab^3\, sin.\, \alpha^2\, cos.\, \alpha}{\frac{4}{3}\, ab\, sin.\, \alpha\, (a^2 + b^2\, cos.\, 2\, \alpha)} = \frac{b^2\, sin.\, 2\, \alpha}{a^2+b^2\, cos.\, 2\, \alpha}.$$

In the rhombus $a = b$, hence

$$tang.\, 2\, \phi = \frac{sin.\, 2\, \alpha}{1 + cos.\, 2\, \alpha} = \frac{2\, sin.\, \alpha\, cos.\, \alpha}{1 + cos.\, \alpha^2 - sin.\, \alpha^2}$$

$$= \frac{2\, sin.\, \alpha\, cos.\, \alpha}{2\, cos.\, \alpha^2} = tang.\, \alpha,$$

therefore $2\, \phi = \alpha$, and $\phi = \frac{\alpha}{2}$. As this angle gives the direction of the diagonal, it follows that the diagonals are free axes of the rhombus.

Example. The sides of the acute angled parallelogram, $ABCD$, Fig. 289, $AB = 2\, a = 16$ inches, and $BC = 2b = 10$ inches, and the angle of the perimeter $ABC = \alpha = 60°$, what directions have its free axis?

$$tang.\, 2\, \phi = \frac{5^2\, .\, sin.\, 120°}{8^2 + 5^2\, .\, cos.\, 120°} = \frac{25\, .\, sin.\, 60°}{64 - 25\, cos.\, 60°} = \frac{25\, .\, 0{,}86603}{64 - 25\, .\, 0{,}5} = 0{,}42040$$

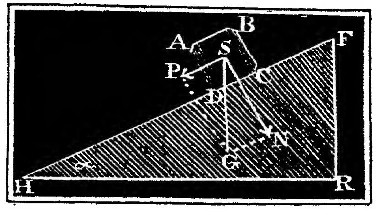
$= tang.\, 22°\, 48'$, or $tang.\, 202°\, 48'$. From this it follows, that $\phi = 11°\, 24'$ and $101°\, 24'$ are the angles of inclination of the two free axes to the side AB. The third free axis stands at right angles to the plane of the parallelogram. These angles determine also the free axes of a right parallelopiped with rhomboidal bases.

CHAPTER III.

OF THE ACTION OF GRAVITY ON MOTIONS ALONG CONSTRAINED PATHS.

§ 238. *Inclined Plane.*—A heavy body may be impeded in various ways from falling freely, and in the following we shall consider only two cases, the one where a body is supported on an inclined plane, and the other where it revolves about a horizontal axis. In both cases the paths of the body are contained in a vertical plane. If the body rests on an inclined plane, its weight may be resolved into two components, of which the one is directed normal to the plane and resisted by it, and the other parallel to the plane, and acts upon the body as a moving force. If G be the weight of the body $ABCD$, Fig. 290, and α the inclination of the inclined plane FHR to the horizon, we shall then have from § 134, for the normal pressure; $N = G \; cos. \; \alpha$, and the moving force $P = G \; sin. \; \alpha$. The motion of the body may be either sliding or rolling, let us next consider the first only. In this case all the parts of the body equally participate in its motion, and hence have a common motion of acceleration p, which is given by the known formula:

Fig. 290.



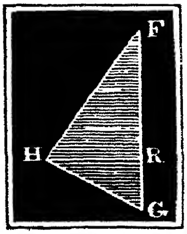
$$p = \frac{\text{force}}{\text{mass}}, = \frac{P}{M} = \frac{G \, sin. \, \alpha}{G} \cdot g = g \; sin. \; \alpha.$$

Therefore $p : g = sin. \; \alpha : 1$, *i. e.* the *accelerated motion of a body on an inclined plane is to the accelerated motion of free descent as the sine of the angle of descent to unity.* In consequence of the friction which takes place, this formula is rarely sufficiently accurate, hence it is necessary in many cases of application to take this into account.

If the body moves on a curved surface, the accelerating force is variable, and at each place equal to the accelerating force, which corresponds with the plane of contact to the curved surface.

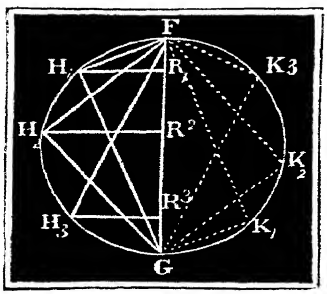
§ 239. A body slides with the initial velocity 0 down an inclined plane, without friction, from § 10 the final velocity after t seconds is: $v = g \; sin. \; \alpha \; . \; t = 32{,}2 \; sin. \; \alpha \; . \; t$ ft., and the space described: $s = \frac{1}{2} g \; sin. \; \alpha \; . \; t^2 = 16{,}1 \; sin. \; \alpha \; . \; t^2$ ft. In free descent $v_1 = gt$, and $s_1 = \frac{1}{2} gt^2$, hence we may put: $v : v_1 = s : s_1 = sin. \; \alpha : 1$, *i. e.* the *final velocity and the space of descent down the inclined plane are to the final velocity and space of free descent as the sine of the angle of inclination of the inclined plane to unity.*

Fig. 291.



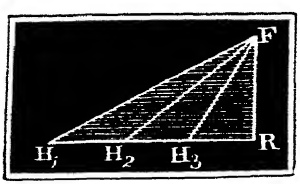
The perpendicular FH of a right-angled triangle, FGH, Fig. 291, with vertical hypothenuse FG, $= FG\ sin.\ FGH = FG\ sin.\ FHR = FG\ sin.\ \alpha$, if α is the angle of inclination of this perpendicular to the horizon, hence $FH : FG = sin.\ \alpha : 1$, and a body describes the vertical hypothenuse FG, and the inclined side FH in one and the same time. The space of free descent corresponding to the space of descent down the inclined plane may be found from this, and the latter from the former by construction. Since the angles of the periphery FH_1G, FH_2G, &c., on the diameter FG, Fig. 292, are right angles, the semicircle on FG cuts off from all the inclined planes, commencing at F, the spaces described with this diameter, and therefore, in equal times, FH_1, FH_2, &c., hence it is asserted, *the chords of a circle and its diameter will descend simultaneously or isochronously.* This isochronism is besides true, not only for the chords FH_1, FH_2, &c., which have their origin at the highest point F of the circle, but also for the chords K_1G, K_2G, &c., which commence at the lowest point G, for chords FK_1, FK_2, &c., may be drawn through F, which have like positions and equal lengths with the chords GH_1, GH_2, &c.

Fig. 292.



§ 240. From the equation $s = \frac{v^2}{2p} = \frac{v^2}{2\,g\,.\,sin.\ \alpha}$, it follows that $s\ sin.\ \alpha = \frac{v^2}{2g}$, and, inversely, $v = \sqrt{2\,gs\ sin.\ \alpha}$. But now $s\ sin.\ \alpha$ is the height FR of the inclined plane or the vertical projection s_1 of the space $FH = s$ upon it, hence the final velocities of bodies which descend with an initial velocity 0 down planes of equal heights F_1H_1, F_2H_2, &c., and of different inclination, Fig. 293, are equal, and also equal to the velocity which a body would acquire if it fell freely from the height FR of these planes.

Fig. 293.



From the equation $s = \frac{1}{2}\,g\ sin.\ \alpha\,.\,t^2$ follows the formula for the time.

$$t = \sqrt{\frac{2\,s}{g\ sin.\ \alpha}} = \frac{1}{sin.\ \alpha}\sqrt{\frac{2\,s\ sin.\ \alpha}{g}} = \frac{1}{sin.\ \alpha}\,.\,\sqrt{\frac{2\,.\,FR}{g}}.$$

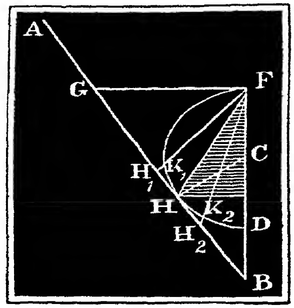
But for a free descent through the height FR the time is:

$t_1 = \sqrt{\frac{2\,FR}{g}}$, it follows accordingly $t : t_1 = 1 : sin.\ \alpha = FH_1 : FR$, *the time of descent down the inclined plane is to the time of free descent*

from the height of this plane as the length of the inclined plane is to its height.

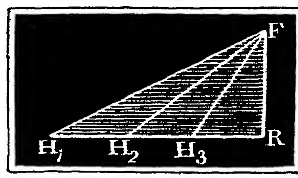
Examples.—1. The initial point F of an inclined plane FH, Fig. 294, is given, and the final point H in a given line AB; required to determine the descent down the plane so that it may take place in the shortest time. If the horizontal line FG be drawn through F to its intersection with AB, and GH be made $= GF$, we shall obtain in H the point sought, and, therefore, in FH the plane of quickest descent; for if through F and H a circle tangent to FG and FH be carried, its isochronously described chords FK_1, FK_2, &c., will be shorter than the lengths FH_1, FH_2, &c., of the corresponding inclined planes; consequently, therefore, the time of descent for these chords will be less than for these lengths, and the time of descent for the inclined plane FH, which coincides with a chord, will be the shortest.

Fig. 294.



2. Required the inclination of that inclined plane FH_1, Fig. 295, down which a body would fall in the same time as if it originally fell freely from the height FR, and then proceeded with the acquired velocity horizontally to H_1. The time of falling down from the vertical height $FR = s_1$ is $t_1 = \sqrt{\frac{2\,s_1}{g}}$, and the acquired velocity at $R: v = \sqrt{2\,gs_1}$. If now no loss of velocity ensue in transition from the vertical to the horizontal motion, which would follow if the corner R were rounded, the space $RH_1 = s_1\, cotg.\, a$ will be uniformly described, and

Fig. 295.



in the time $t_2 = \frac{s_1\, cotg.\, a}{v} = \frac{s_1\, cotg.\, a}{\sqrt{2\,gs_1}} = \frac{1}{2}\, cotg.\, a \sqrt{\frac{2\,s_1}{g}}$. The time of descent down the inclined plane is $t = \frac{1}{sin.\, a} \sqrt{\frac{2\,s_1}{g}}$; hence, if we put $t = t_1 + t_2$, we shall obtain the equation of condition $\frac{1}{sin.\, a} = 1 + \frac{1}{2}\, cotg.\, a$, whose solution will give $tang.\, a = \frac{3}{4}$. In the corresponding inclined plane, accordingly, the height is to the base and to the length as 3 is to 4 is to 5, and the angle of inclination is $a = 36°\ 52'\ 11''$.—3. The time for sliding down an inclined plane of a given base a is

$$t = \sqrt{\frac{2\,s}{g\ sin.\, a}} = \sqrt{\frac{2\,a}{g\ sin.\, a\ cos.\, a}} = \sqrt{\frac{4\,a}{g\ sin.\, 2\,a}};$$

hence the descent is quickest when $sin.\, 2\,a$ is a maximum, *i. e.* $= 1$; therefore $2\,a° = 90°$, or $a° = 45°$. Hence, water falls down in the shortest time from roofs of 45° inclination.

§ 241. If the motion on an inclined plane proceeds with a certain initial velocity c, we shall then have to apply the formulæ found in § 13 and § 14. According to these the terminal velocity of a body ascending an inclined plane is $v = c - g\ sin.\, a\ .\ t$, and the space described $s = ct - \frac{1}{2}\, g\ sin.\, a\ .\ t^2$; on the other hand, for a body falling down the inclined plane:

$$v = c + g\ sin.\, a\ .\ t, \text{ and } s = ct + \tfrac{1}{2}\, g\ sin.\, a\ .\ t^2.$$

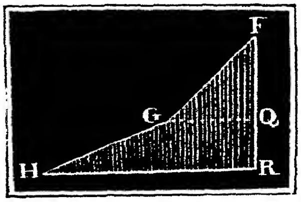
In both cases of motion the formula is true:

$$s = \frac{v^2 - c^2}{2\,g\ sin.\, a}, \text{ or } s\ sin.\, a = \frac{v^2 - c^2}{2\,g} = \frac{v^2}{2g} - \frac{c^2}{2g}.$$

The vertical projection, therefore, (s sin. a) of the space (s) described along the inclined plane is always equal to the difference of the heights due to the velocity.

If two inclined planes FGQ and GHR, Fig. 296, meet each other in a rounded edge, no impulse will take place in the passage from one plane to the other, and for this reason, no loss of velocity ensue; the rule for the descent of a body down this combination of two planes is also true, *the height of descent* (FR) *is equal to the difference of the heights due to the velocity.* It is easy to ascertain that this rule is correct also for the ascent or descent on any system of any number of planes, and for the ascent or descent on curved lines or surfaces. (Compare § 82.)

Fig. 296.



Examples.—1. A body ascends with a 21 feet initial velocity an inclined plane of 22° inclination, what is the amount of its velocity and its space described in $1\frac{1}{2}$ seconds? The velocity is:
$v = 21 - 32{,}2\ sin.\ 22^\circ\ .\ 1{,}5 = 21 - 32{,}2\ .\ 0{,}3746\ .\ 1{,}5 = 2{,}906$ feet; and the space:

$$s = \frac{c+v}{2}.t = \frac{21 + 2{,}906}{2}\ .\ \frac{3}{2} = \frac{23{,}906\ .\ 3}{4} = 17{,}928 \text{ feet.}$$

2. How high does a body, with an initial velocity of 36 feet, ascend an inclined plane of 48° acclivity? The vertical height is $s_1 = \frac{v^2}{2g} = 0{,}01550\ v^2 = 0{,}0155\ .\ 36^2 = 21{,}638$ feet; hence the whole space up the inclined plane: $s = \frac{s_1}{sin.\ a} = \frac{21.638}{sin.\ 48^\circ} = 28{,}494$ feet. The time required is:

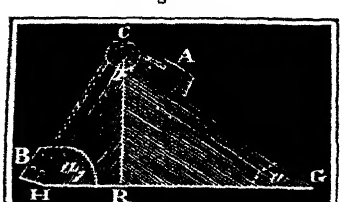
$$t = \frac{2\ .\ s}{v} = \frac{2\ .\ 28{,}499}{36} = \frac{28{,}494}{18} = 1{,}583 \text{ seconds.}$$

§ 242. Sliding friction exerts a considerable influence upon the ascent and descent of a body along an inclined plane. From the weight G of the body, and from the angle of inclination a of the inclined plane, the normal pressure follows, $N = G\ cos.\ a$, and again from this, the friction $F = f\ N = f\ G\ cos.\ a$. If we subtract this from the force $P = G\ sin.\ a$, with which gravity urges the body down the plane, there then remains for the moving force $= G\ sin.\ a - f\ G\ cos.\ a$, and the accelerating force of the body sliding down the plane is known:

$$p = \frac{\text{the force}}{\text{the mass}} = \frac{G\ sin.\ a - f\ G\ cos.\ a}{G}\ g = (sin.\ a - f\ cos.\ a)\ g.$$

The moving force of the body ascending the inclined plane is negative and $= G\ sin.\ a + f\ .\ G\ cos.\ a$, hence also the accelerating force p is negative and $= -(sin.\ a + f\ cos.\ a)\ g$.

Fig. 297.



If two bodies are supported on different planes FG and FH, Fig. 297, by perfectly flexible strings connected with each other, passing over a roller C, it is then possible for one of the two bodies to descend and pull up the other. If we represent the weights of these bodies by G and G_1, and the angles of inclination of the inclined planes along which they move by

α and α_1, and if we assume that G descends and draws G_1 upwards, we shall then obtain as the moving force:
$G\ sin.\ \alpha - G_1\ sin.\ \alpha_1 - f\ G\ cos.\ \alpha - f\ G_1\ cos.\ \alpha_1 = G\ (sin.\ \alpha - f\ cos.\ \alpha) - G_1\ (sin.\ \alpha_1 + f\ cos.\ \alpha_1)$, and the mass moved $= \frac{G + G_1}{g}$, hence the accelerated motion with which G descends and G_1 ascends:

$$p = \frac{G\ (sin.\ \alpha - f\ cos.\ \alpha) - G_1\ (sin.\ \alpha_1 + f\ cos.\ \alpha_1)}{G + G_1} \cdot g.$$

Since friction as a resisting force can generate no motion, it is requisite for the fall of G and the rise of G_1, that

$G\ (sin.\ \alpha - f\ cos.\ \alpha)$ be $> G_1\ (sin.\ \alpha_1 + f\ cos.\ \alpha_1)$, therefore $\frac{G}{G_1} > \frac{sin.\ \alpha_1 + f\ cos.\ \alpha_1}{sin.\ \alpha - f\ cos.\ \alpha}$. If, on the other hand, G_1 descend, and G be drawn up, then must:

$$\frac{G_1}{G} \text{ be} > \frac{sin.\ \alpha + f\ cos.\ \alpha}{sin.\ \alpha_1 - f\ cos.\ \alpha_1}, \text{ or, } \frac{G}{G_1} < \frac{sin.\ \alpha_1 - f\ cos.\ \alpha_1}{sin.\ \alpha + f\ cos.\ \alpha}.$$

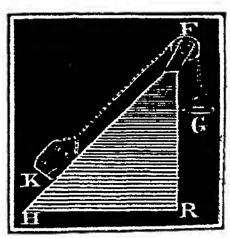
So long, however, as $\frac{G}{G_1}$ lies within the limits:

$$\frac{sin\ \alpha_1 + f\ cos.\ \alpha_1}{sin.\ \alpha - f\ cos.\ \alpha} \text{ and } \frac{sin\ \alpha_1 - f\ cos.\ \alpha_1}{sin.\ \alpha + f\ cos.\ \alpha},$$

so long will the friction resist motion.

Examples.—1. A sledge moves down an inclined snow plane, 150 feet long and 20° inclination, and when arrived at the bottom, proceeds along a horizontal one until friction brings it to rest. If the co-efficient of friction between the snow and the sledge be taken $= 0{,}03$ feet, what space will the sledge describe along the horizontal plane, neglecting the resistance of the air? The accelerating force $p = (sin.\ \alpha - f\ cos.\ \alpha)\ g = (sin.\ 20°) \cdot 32{,}2 = (0{,}3420 - 0{,}03 \cdot 0{,}9397) \cdot 32{,}2 = 0{,}3138 \cdot 32{,}2 = 10{,}104$ feet; hence, the final velocity of descent is,
$v = \sqrt{2\ ps} = \sqrt{2 \cdot 10{,}104 \cdot 150} = \sqrt{3031{,}2} = 55{,}54$ feet. On the horizontal plane the accelerating force is $p_1 = -fg = -0{,}03 \cdot 32{,}2 = 0{,}966$ feet; hence, the space $s_1 = \frac{v^2}{2\ f\ g} = \frac{3031{,}2}{1{,}932} = 1630$ feet. The time of descent is $t = \frac{2\ s}{v} = \frac{300}{55{,}54} = 5{,}22$ seconds, and that for sliding onward $t_1 = \frac{2\ s_1}{v} = \frac{3260}{55{,}54} = 58{,}6$ seconds; hence the whole time of the course $t + t_1 = 63{,}82$ seconds $= 1'\ 3{,}82''$.—2. A filled tub K, Fig. 298. of 250 lbs. clear weight, is drawn up an inclined plane FH, 70 feet long and of 50° inclination, by a descending weight G of 260 lbs.; what will be the time required for this, if the co-efficient of friction of the tub along its path amount to 0,36. The moving force is $= G - (sin.\ \alpha + f\ cos.\ \alpha)\ K = 260 - (sin.\ 50° + 0{,}36 \cdot cos.\ 50°) \cdot 250 = 260 - 0{,}9974 \cdot 250 = 10{,}6$ lbs.; hence, the accelerating force $p = \frac{10{,}6}{250 + 260} = \frac{10{,}6}{510} = 0{,}0208$ lbs.; further, the time

Fig. 298.



$$t = \sqrt{\frac{2\ s}{p}} = \sqrt{\frac{140}{0{,}0208}} = \sqrt{6731} = 82{,}04 \text{ seconds} = 1'\ 22'',$$

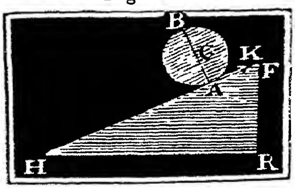
and the final velocity $v = \frac{2\ s}{t} = \frac{140}{82} = 1{,}70$ feet.

§ 243. *Rolling Motion.*—When a carriage rolls down an inclined plane, the friction of the axle chiefly acts in opposition to the accelerating force; if r be the radius of the axle, and a that of the wheel,

the friction will amount to $\frac{fr}{a} N = \frac{fr}{a} G \cos. \alpha$, and hence the accelerating force $p = (\sin. \alpha - \frac{fr}{a} \cos. \alpha)\, G$.

If a round body AB, a cylinder or sphere, for example, roll down an inclined plane FH, Fig. 299, we have to consider a progressive and a rotatory motion at the same time. Generally the acceleration of the progression is equal to that of the rotation (§ 156); since if we put the moment of inertia of the rolling body $= G y^2$, and the radius of the cylinder $= a$, we shall then obtain for the force $AK = K$, with which the cylinder is set into revolution by virtue of the penetration of its parts into those of the inclined plane:* $K = p \cdot \frac{G y^2}{g a^2}$.

Fig. 299.



But the force K acts opposed to the force of descent $G \sin. \alpha$, hence it follows that the moving force for progressive motion $= G \sin. \alpha - K$, and the accelerating force $p = \frac{G \sin. \alpha - K}{G} \cdot g$. If we eliminate K from both equations, we shall obtain $G p = G g \sin. \alpha - \frac{G y^2}{a^2} \cdot p$, consequently the accelerating force sought:

$$p = \frac{g \sin. \alpha}{1 + \frac{y^2}{a^2}}.$$

For the case of a homogeneous rolling cylinder $y^2 = \frac{1}{2} a^2$ (§ 221), hence $p = \frac{g \sin. \alpha}{1 + \frac{1}{2}} = \frac{2}{3} g \sin. \alpha$; but for a sphere $y^2 = \frac{2}{5} a^2$ (§ 222), hence $p = \frac{g \sin. \alpha}{1 + \frac{2}{5}} = \frac{5}{7} g \sin. \alpha$; therefore, the accelerating force of the rolling cylinder is only $\frac{2}{3}$, that of a rolling sphere only $\frac{5}{7}$ that of a body sliding without friction.

The force of rotation is:

$$K = \frac{g \sin. \alpha}{1 + \frac{y^2}{a^2}} \cdot \frac{G y^2}{g a^2} = \frac{G y^2 \sin. \alpha}{a^2 + y^2}.$$

As long as this is less than the sliding friction $f\, G \cos. \alpha$, the body descends rolling perfectly down the plane. But if

$$K \text{ is} > f\, G \cos. \alpha, \text{ i. e. } \tan g.\, \alpha > f\left(1 + \frac{a^2}{y^2}\right),$$

the friction is no longer sufficient to communicate to the body a velocity of rotation equal to its velocity of progression; hence the acceleration of progression, as for sliding friction, is:

* See above, pages, 169 to 175, note.

$$p = \frac{G \sin. \alpha - f G \cos. \alpha}{G} . g = (\sin. \alpha - f \cos. \alpha) g,$$

and that of rotation:

$$p_1 = \frac{f G \cos. \alpha}{G y^2 \div a^2} . g = f \frac{a^2}{y^2} g \cos. \alpha.$$

For a carriage of the weight G with wheels of the radius a, and with the moment of inertia $G_1 y^2$, we have:

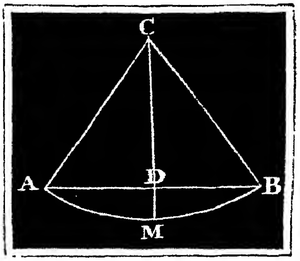
$$K = p \frac{G_1 y^2}{g a^2} \text{ and } p = \frac{G \sin. \alpha - f \frac{r}{a} G \cos. \alpha - K}{G} . g, \text{ i. e.}$$

$$p = \frac{g (\sin. \alpha - f \frac{r}{a} \cos. \alpha)}{1 + \frac{G_1 y^2}{G a^2}}.$$

Examples.—1. A loaded wagon of 3600 lbs. weight, with wheels 4 feet high, and moment of inertia 2000 ft. lbs., rolls down an inclined plane of 12° inclination, what will be its accelerated motion, if the co-efficient of axle friction $= 0,15$, and the thickness of the axles of the wheels amounts to 3 inches? It is $\frac{G_1 y^2}{G a^2} = \frac{2000}{3600 . 2^2} = \frac{5}{36} = 0,139$, and $f \frac{r}{a} = 0,15 . \frac{1}{4 . 4} = 0,0094$ hence the accelerating force sought is $p = \frac{32,2 (\sin. 12^\circ - 0,0094 . \cos. 12^\circ)}{1 + 0,139} = \frac{32,2 (0,2079 - 0,0094 . 0,978)}{1,139} = \frac{32,2 . 0,1987}{1,139} = 6,398$ feet.—2. What will be the accelerating forces of a solid cylinder rolling down an inclined plane of a 40° angle of descent? The co-efficient of the sliding friction of the cylinder on the plane $= 0,24$, we have then $f \left(1 + \frac{a^2}{y^2}\right) = 0,24 (1 + 2) = 0,72$; but now the *tang.* $40^\circ = 0,839$, hence the *tang.* α is greater than $f \left(1 + \frac{a^2}{y^2}\right)$, and the acceleration of the rolling motion less than that of the progressive. The last is $p = (\sin. \alpha - f \cos. \alpha) g = (0,6428 - 0,24 \cdot 0,7660) . 32,2 = 0,459 . 32,2 = 14,78$ feet, but the first only $p_1 = 0,24 . 2 . 32,2 \cos. 40^\circ = 11,85$ feet.

§ 244. *Circular Pendulum.*—Equilibrium subsists in a body suspended to a horizontal axis so long as its centre of gravity lies vertically below the axis; but if its centre of gravity be drawn out of the vertical plane containing the axis, and the body be left to itself, it will take an oscillatory motion; that is, it will move up and down in a circle. In general, however, a body oscillating about a horizontal axis is called a *pendulum.* If the oscillating body is a material point, and its connection with the axis of revolution be made by a line devoid of weight, we then have the *mathematical* or *simple pendulum;* but if the pendulum consists of a body having dimensions, or of several bodies, we have then a *compound, physical,* or *material pendulum.* Such a pendulum may be regarded as a connection of simple pendulums oscillating about a common axis. The simple pendulum is an imaginary one only, but its assumption possesses great advantage, because it is easy to reduce the theory of the motion of the compound to that of the simple pendulum.

Fig. 300.

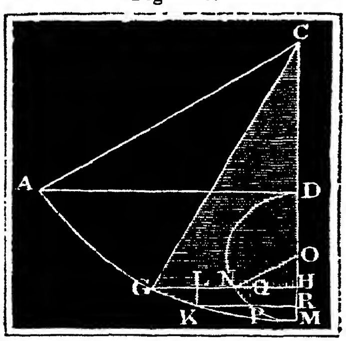


If the pendulum suspended at C, Fig. 300, be drawn out of its vertical position CM into the position CA, and then left to itself, it will go back by virtue of its gravity with an accelerated motion towards CM, and its mass will arrive at its lowest point M with a velocity v, whose height $\frac{v^2}{2g}$ is equal to the height of descent DM. In virtue of this velocity, it will now describe on the other side the arc $MB = MA$, and will thereby ascend to the height DM. From B it will again fall back to M and A, and so it will go on successively describing the circular arc AB. If the resistance of the air and friction were entirely set aside, this oscillation of the pendulum would go on indefinitely; but because these resistances can never be done away with, the amplitude of the oscillation will become smaller and smaller, and the pendulum come at last to a state of rest.

The motion of the pendulum from A to B is called an *oscillation*, the arc AB the *amplitude*, the angle measuring half the amplitude by which the pendulum is distant from either side of the vertical CM, *the angle of elongation* or *angle of deviation*. Lastly, the time in which the pendulum makes an oscillation, is called the *time of oscillation.*

§ 245. On account of the frequent application of the pendulum to the purposes of life, to clocks namely, it is of consequence to know the times of oscillation, hence the determination of these is one of the principal problems in mechanics. With the view of solving this problem, let us put the length of the pendulum $AC = MC = r$, Fig. 301, and the height of ascent or descent corresponding to a complete oscillation $MD = h$. Let us assume that the pendulum has fallen from A to G, and let the height of fall $DH = x$ correspond to this motion, we may then put the acquired velocity $v = \sqrt{2gx}$,

Fig. 301.



and the particle of time in which the particle of space GK is described, $\tau = \frac{GK}{v} = \frac{GK}{\sqrt{2gx}}$. If, now, from the centre O of $MD = h$ and the radius $OM = OD = \frac{1}{2}h$, we describe the semicircle MND, we then have a portion of this arc NP of the height $PQ = KL = RH$ equal to GK, which is in a simple ratio to this particle of space GK. From the similarity of the triangles GKL and CGH, $\frac{GK}{KL} = \frac{CG}{GH}$, and from the similarity of the triangles NPQ and ONH,

$\frac{NP}{PQ} = \frac{ON}{NH}$; hence, if we divide these two equations by each other, and bear in mind that $KL = PQ$, we then obtain the ratio of the said portion of arc: $\frac{GK}{NP} = \frac{CG \cdot NH}{GH \cdot ON}$. From the properties of the circle, and from the theorem of the mean proportional, $\overline{GH}_2 = MH\,(2\,CM - MH)$ and $\overline{NH}^2 = MH \cdot DH$; hence it follows:

$$\frac{GK}{NP} = \frac{CG \cdot \sqrt{DH}}{ON \cdot \sqrt{2\,CM - MH}} = \frac{r\sqrt{x}}{\frac{1}{2}\,h\sqrt{2\,r - (h-x)}},$$

and the time for describing an element of space is:

$$\tau = \frac{r\sqrt{x}}{\frac{1}{2}\,h\sqrt{2\,r - (h-x)}} \cdot \frac{NP}{\sqrt{2\,gx}} = \frac{2\,r}{h\sqrt{2\,g\,[2\,r - (h-x)]}} \cdot NP$$
$$= \sqrt{\frac{r}{g}} \cdot \frac{NP}{h\sqrt{1 - \frac{h-x}{2r}}}.$$

In most cases of application, a small angle of deviation is given to the pendulum, and for this reason $\frac{h}{2r}$, as also $\frac{x}{2r}$, and, therefore, also $\frac{h-x}{2r}$ is so small a quantity that we may neglect it as well as its powers, and now put $\tau = \sqrt{\frac{r}{g}} \cdot \frac{NP}{h}$. The duration of a semi-oscillation, or the time in which the pendulum describes the arc AM, is equal to the sum of all the particles of time corresponding to the elements GK or NP, or as $\frac{1}{h} \cdot \sqrt{\frac{r}{g}}$ is a constant factor, equal to $\frac{1}{h}\sqrt{\frac{r}{g}}$ times the sum of all the elements forming the semicircle DNM; *i. e.* $= \frac{1}{h}\sqrt{\frac{r}{g}}$ times the semicircle $\left(\frac{\pi h}{2}\right)$ itself, therefore

$$= \frac{1}{h}\sqrt{\frac{r}{g}} \cdot \frac{\pi h}{2} = \frac{\pi}{2}\sqrt{\frac{r}{g}}.$$

The pendulum, however, requires the same time for ascending, for here the velocities are the same, and only opposite in direction, and for this reason the duration of a complete oscillation is twice as great,

$$i.\,e.\ t = \pi\sqrt{\frac{r}{g}}.$$

§ 246. To determine the duration of an oscillation with greater accuracy, which is necessary where the angles of oscillation are large, let us transform the expression:

$\sqrt{\dfrac{1}{1-\dfrac{h-x}{2r}}} = \left(1-\dfrac{h-x}{2r}\right)^{-\frac{1}{2}}$ into the series

$$1+\tfrac{1}{2}\cdot\frac{h-x}{2r}+\tfrac{1}{8}\cdot\left(\frac{h-x}{2r}\right)^2+\ldots,$$

and we shall obtain the time for an element of space

$$\tau=\left[1+\tfrac{1}{2}\cdot\frac{h-x}{2r}+\tfrac{1}{8}\left(\frac{h-x}{2r}\right)^2+\ldots\right]\sqrt{\frac{r}{g}}\cdot\frac{NP}{h}.$$

If we put the angle NOM, subtended at the centre by NM, $=\phi$, we shall then also obtain

$$MH=h-x=NO\,(1-\cos.\phi)=\tfrac{1}{2}h\,(1-\cos.\phi);\text{ hence:}$$

$$\tau=\left(1+\tfrac{1}{2}\cdot\frac{h\,(1-\cos.\phi)}{4r}+\ldots\right)\sqrt{\frac{r}{g}}\cdot\frac{NP}{h}.$$

If we divide the semicircle DNM into n equal parts, and if we put each $= NP = \dfrac{\pi h}{2n}$, we shall obtain

$$\tau=\left(1+\tfrac{1}{2}\cdot\frac{h\,(1-\cos.\phi)}{4r}+\ldots\right)\sqrt{\frac{r}{g}}\cdot\frac{\pi}{2n};$$

by substituting successively for $\phi=\dfrac{\pi}{n},\dfrac{2\pi}{n},\dfrac{3\pi}{n}\ldots$ to $\dfrac{n\pi}{n}$, and adding the results, we shall then obtain half the time of an oscillation:

$$t=\left(n+\frac{h}{8r}(n-\text{the sum of all the cosines})+\ldots\right)\sqrt{\frac{r}{g}}\cdot\frac{\pi}{2n}.$$

But the sum of the cosines of all the angles from $\phi=0$ to $\phi=\pi$ is $=0$; hence, we have more correctly : $t=\left(1+\dfrac{h}{8r}\right)\cdot\pi\sqrt{\dfrac{r}{g}}$.

If we have regard to more members of the series, we shall obtain:

$$t=\left[1+(\tfrac{1}{2})^2\cdot\frac{h}{2r}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{h}{2r}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2\cdot\left(\frac{h}{2r}\right)^3+\ldots\right]\cdot\pi\sqrt{\frac{r}{g}},$$

the last formula but one is, however, generally sufficient. If the pendulum oscillates in a semicircle, we then have $h=r$; hence the duration of an oscillation:

$$t=\left(1+\tfrac{1}{8}+\frac{9}{256}+\frac{225}{18432}+\ldots\right)\pi\sqrt{\frac{r}{g}}=1{,}180\ldots\pi\sqrt{\frac{r}{g}}.$$

From the angle of elongation α, it follows that

$\cos.\alpha=\dfrac{r-h}{r}=1-\dfrac{h}{r}$, therefore, $\dfrac{h}{r}=1-\cos.\alpha$; and hence, $\dfrac{h}{8r}$ $=\tfrac{1}{4}\cdot\dfrac{1-\cos.\alpha}{2}=\tfrac{1}{4}\left(\sin.\dfrac{\alpha}{2}\right)^2$; from this, consequently, the correction for the time of oscillation corresponding to a given angle of elongation may be found. If, for example, this angle $=15^0$, we have $\dfrac{h}{8r}=\tfrac{1}{4}\left(\sin.\dfrac{15^0}{2}\right)^2=0{,}00426$; on the other hand, for $\alpha=5^0$: $\dfrac{h}{8r}$

$= 0{,}00047$; for the latter angle of elongation, therefore, the time of oscillation is $t = 1{,}00047 \, . \, \pi \sqrt{\frac{r}{g}}$.

We may, therefore, for a deviation under 5°, put tolerably accurately the time of oscillation:

$$t = \pi \sqrt{\frac{r}{g}} = \frac{\pi}{\sqrt{g}} \sqrt{r} = 0{,}562 \sqrt{r}.$$

§ 247. As the angle of deviation does not appear in the formula $t = \pi \sqrt{\frac{r}{g}}$, it follows that the small times of oscillation of pendulums are independent of this angle, and therefore that pendulums of equal length, but of different angles of deviation, vibrate isochronously, or perform their oscillations in equal times. A pendulum deviating 4° has the same time of oscillation as a pendulum deviating 1°.

If we compare the time of oscillation t with the time of free descent, we shall then arrive at the following. The time of free descent from the height r will be t_1

$$= \sqrt{\frac{2r}{g}} = \sqrt{2} \, . \sqrt{\frac{r}{g}}, \text{ hence } t : t_1 = \pi : \sqrt{2};$$

the time of an oscillation is, therefore, to the time in which a body of the length of the pendulum freely descends, as π to the square root of 2, or since t_1 is also $= \sqrt{\frac{4 \, . \, \frac{1}{2} r}{g}} = 2 \sqrt{\frac{\frac{1}{2} r}{g}}$, the *time of oscillation is to the time of descent of half the length of the pendulum as π is to 2.*

If we put the times of oscillation t and t_1, corresponding to the lengths of the pendulum r and r_1, we then obtain $t : t_1 = \sqrt{r} : \sqrt{r_1}$; therefore, *for one and the same acceleration of gravity, the times of oscillation are as the square roots of the lengths of the pendulum.* On the other hand, if n be the number of oscillations which a pendulum makes in a certain time, one minute, and n_1 the number which another pendulum makes in the same time, we then have $t : t_1 = \frac{1}{n} : \frac{1}{n_1}$, hence, inversely, $n : n_1 = \sqrt{r_1} : \sqrt{r}$, i. e. *the number of oscillations is in an inverse ratio to the square roots of the lengths of pendulums.* A pendulum four times the length gives, therefore, half the number of oscillations.

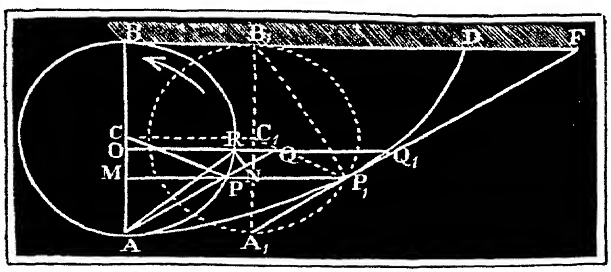
A pendulum is called a *seconds pendulum*, when its time of oscillation is one second. If we put $t = 1$ into the formula $t = \pi \sqrt{\frac{r}{g}}$, we obtain the length of the seconds pendulum $r = \frac{g}{\pi^2} = 39{,}13929$ inches $= 0{,}9938$ meters.

From the formula $t = \pi \sqrt{\frac{r}{g}}$ it follows by inversion that $g = \left(\frac{\pi}{t}\right)^2 r$; the acceleration of gravity may be found, therefore, from the length of a pendulum, and from its time of oscillation t. This method is both simpler and safer than that of Attwood's machine.

Remark. The diminution of gravity from the poles to the equator has been proved by pendulum observations, and its quantity determined. This diminution is due to the effect of the centrifugal force, which is generated by the diurnal rotation of the earth about its axis, and to the increase of the earth's radius from the poles to the equator. The centrifugal force at the equator diminishes gravity by $\frac{1}{290}$ of its value (§ 231), whilst at the poles it is null. If β be the latitude of the place of observation, the accelerating force of gravity from pendulum observations will be $g = 32{,}2\,(1 - 0{,}00259\, \cos.\, 2\,\beta)$, therefore at the equator where $\beta = 0$; therefore $\cos.\, 2\,\beta = 1$, $g = 32{,}2\,(1 - 0{,}00259) = 32{,}11$ feet, and at the poles, where $\beta = 90°$; therefore $\cos.\, 2\,\beta = \cos.\, 180° = -1$; $g = 32{,}2 \,.\, 1{,}00259 = 32{,}283$ ft. For the rest g is less on mountains and in mines than at the level of the sea.

§ 248. *Cycloid.*—We may in an infinite number of ways set a body into vibration, or into an oscillating motion, and we call every body in this condition of motion a *pendulum*, and distinguish accordingly several kinds of pendulums, for example, the *circular pendulum*, which we have already considered, and the *cycloidal*, where the body, by virtue of gravity, oscillates to and fro in a cycloidal arc, and the *torsion pendulum*, where the body vibrates by virtue of the torsion of a thread, or wire, &c. We shall here speak only of the *cycloidal pendulum*.

Fig. 302.



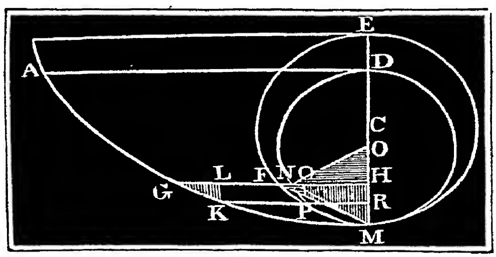
The *cycloid AD*, Fig. 302, is a curved line described by a point A of a circle APB which rolls along a straight line BD. If this generating circle has rolled forward $BB_1 = CC_1$, and, therefore, come into the position A_1B_1, it has then also revolved through the arc $AP = A_1P_1 = BB_1 = PP_1$, consequently the ordinate corresponding to any absciss MP_1 = ordinate MP of the circle plus the arc of revolution AP. In this rolling the generating circle revolves about the point of contact at each instant with the base line, if, therefore, it be in A_1B_1, it will then revolve about B_1, and describe thereby the elementary arc P_1Q_1 of the cycloid; consequently the chord B_1P_1 will be the direction of the normal, and the chord A_1P_1 that of the tangent to the cycloid at the point P_1. The prolongation PQ of the chord AP reaching to the ordinate OQ_1 is, therefore, equal to the element of the cycloid P_1Q_1, as further the space of revolution is equal to the space RQ of

progression, PQ is then the base line of an isosceles triangle PRQ, and equal to double the line PN, which the perpendicular RN cuts off, but PN is the difference of the two contiguous chords AP, AR, and consequently the element of the cycloid P_1Q_1 = twice the difference of the chords ($AR—AP$).

As the continuous elementary arcs make up together the whole arc AP_1, and likewise the aggregate of the differences of the chords, the whole chord AP, the length of the cycloidal arc AP_1, is, from this, equal to double the chord of the circle AP, appertaining to it. To the semi-cycloid AP_1D, corresponds the diameter as a chord of a circle, hence the length of the half of the cycloid is equal to double the diameter of the generating circle.

§ 249. *Cycloidal Pendulum.*—From the above known properties of the cycloid, the theory of the cycloidal pendulum, or the formula for the time of oscillation of a body vibrating in a cycloidal arc, may be easily developed. Let AKM, Fig. 303, be the half of the cycloidal arc in which a body ascends or descends, or oscillates, and ME the generating circle, therefore, $CE = CM = r$ its radius.

Fig. 303.



If the body has described the arc AG, it has, therefore, fallen from the height $DH = x$ (§ 246), it has then acquired the velocity $v = \sqrt{2gx}$, with which it describes the elementary arc CK in the time $\tau = \frac{GK}{v} = \frac{GK}{\sqrt{2gx}}$.

But from the similarity of the triangles GLK and FHM, $\frac{GK}{KL} = \frac{FM}{MH}$, or as $\overline{FM}^2 = MH \cdot ME$, $\frac{GK}{KL} = \frac{\sqrt{MH \cdot ME}}{MH} = \frac{\sqrt{ME}}{\sqrt{MH}}$; from the similarity of the triangles NPQ and ONH, $\frac{NP}{PQ} = \frac{ON}{NH}$, or since $\overline{NH}^2 = MH \cdot DH$, $\frac{NP}{PQ} = \frac{ON}{\sqrt{MH \cdot DH}}$. Now $KL = PQ$, hence it follows by division:

$$\frac{GK}{NP} = \frac{\sqrt{ME}}{\sqrt{MH}} \cdot \frac{\sqrt{MH \cdot DH}}{ON} = \frac{\sqrt{ME \cdot DH}}{ON},$$

or, since ON is half the height of descent $= \frac{h}{2}$, $ME = 2r$, and $DH = x$:

$$\frac{GK}{NP} = \frac{\sqrt{2rx}}{\frac{1}{2}h} = \frac{2\sqrt{2rx}}{h}.$$

By putting $GK = \frac{2\sqrt{2rx}}{h} \cdot NP$ into the formula $\tau = \frac{GK}{\sqrt{2gx}}$ we obtain:

$$\tau = \frac{2\sqrt{2rx}}{\sqrt{2gx} \cdot h} \cdot NP = \frac{2}{h}\sqrt{\frac{r}{g}} \cdot NP.$$

The time of falling from A to M is the sum of all the values of τ, which are obtained; if for NP all the particles of the semicircle DNM be successively substituted, therefore, $= \frac{2}{h}\sqrt{\frac{r}{g}}$ times the semicircle $DNM \left(\frac{\pi}{2} h\right)$. In this manner we obtain the time for falling through the arc AM,

$$= \frac{\pi}{h} h \cdot \frac{2}{h}\sqrt{\frac{r}{g}} = \pi\sqrt{\frac{r}{g}},$$

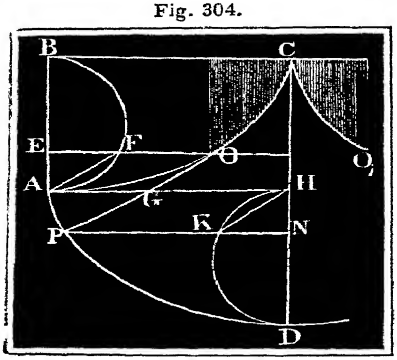
and as the time for ascending the arc MB is likewise as great, the time of oscillation, or the time of describing the whole arc AMB:

$$t = 2\pi\sqrt{\frac{r}{g}} = \pi\sqrt{\frac{4r}{g}}.$$

As this quantity is quite independent of the length of the arc, it follows that, mathematically speaking, the times of oscillation for all arcs of one and the same cycloid are equal, the oscillations of the cycloidal pendulum are, therefore, perfectly isochronous. If we compare this formula with the formula for the time of oscillation of a circular pendulum, it follows that the times of oscillation for both kinds of pendulums are equal, if the length of the circular pendulum is equal to four times the radius of the generating circle of the cycloidal pendulum.

Remark 1. It may be proved by the higher calculus that the cycloid has, besides the property of *isochronism* or *tautochronism*, also that of *brachistochronism*, which is that line between two given points in which a body falls in the shortest time from one point to the other.

Fig. 304.

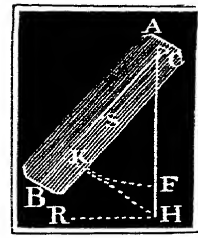


Remark 2. In order to make a body, suspended to a perfectly flexible thread, vibrate in a cycloidal arc, and thereby represent the cycloidal pendulum, we suspend the body between two cycloidal arcs CO and CO_1, Fig. 304, so that the thread for every deviation unwinds from the one arc and winds round the other. By this winding and unwinding of the thread COP, its extremity P describes a curve similar to the given cycloid, and it may be similarly represented that the evolute of the cycloid is a similar cycloid in an inverse position. As the length of half the cycloid $COA = CD = 2\,AB$, we have likewise the arc = to the straight line evolved OP; but the arc $OA = 2$ chord $AF = 2\,GO$, hence also $PG = GO = AF$, and $HN = AE$. If now we describe upon DH a semicircle DKH and draw the ordinate NP, we then have $KH = PG$; and hence also $PK = GH = AH - AG = AH - FO =$ arc AFB — arc AF = arc BF = arc DK; and lastly, the ordinate NP = the ordinate NK of the circle, plus the corresponding arc DK; therefore NP is the ordinate of a cycloid, and DPA the cycloid corresponding to the generating circle DKH.

For the application of the cycloidal pendulum to clocks, see *Jahrbücher des polytech.* Institutes in Wien., vol. xx. art. 2.

§ 250. *Compound Pendulum.*—To find the time of oscillation of the compound pendulum, or that of any other body AB, Fig. 305, oscillating about a horizontal axis C, let us first seek the centre of oscillation, *i. e.* that point K of the body, which if it oscillates of itself about C, or forms a mathematical pendulum, has the same time of oscillation as the whole body. It is easily seen from this explanation that there are several centres of oscillation in a body, but in general, that point only is meant which lies with the centre of gravity, in one and the same perpendicular to the axis of revolution.

Fig. 305.



From the variable angle of deviation $KCF = \phi$, the accelerating force of the isolated point K, $= g\ sin.\ \phi$, because we may suppose that it slides down an inclined plane of the inclination $KHR = KCF$. But if My^2 be the moment of inertia of the entire body or set of bodies AB, Ms will be its statical moment, *i. e.* the product of the mass, and the distance $CS = s$ of its centre of gravity S from the axis of revolution C, and r the distance CK of the centre of oscillation K from the axis of revolution, or the length of the simple pendulum which vibrates isochronously with the material pendulum AB, we have then the mass reduced to $K = \frac{My^2}{r^2}$, and the force of revolution reduced to this $\frac{s}{r}\ M g\ sin.\ \phi$; consequently the accelerating force $= \frac{\text{force}}{\text{mass}} = \frac{s}{r} M g\ sin. \phi \div \frac{My^2}{r^2} = \frac{Msr}{My^2} \cdot g\ sin.\ \phi$. That this pendulum may have the same time of oscillation as a mathematical one, it is requisite that both should have their motion in every position equally accelerated, that therefore, $\frac{Msr}{My^2} \cdot g\ sin.\ \phi = g\ sin.\ \phi$. Now this equation gives:

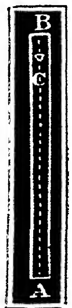
$$r = \frac{My^2}{Ms} = \frac{\text{moment of inertia}}{\text{statical moment}}.$$

We, therefore, find the distance of the centre of oscillation from the centre of gyration, or the length of the simple pendulum, which has a time of oscillation equal to that of the compound one, if we divide the moment of inertia of the compound pendulum by its statical moment.

If we substitute this value in the formula $t = \pi \sqrt{\frac{r}{g}}$, we obtain for the time of oscillation of the compound pendulum the formula $t = \pi \sqrt{\frac{M\,y^2}{M g s}} = \pi \sqrt{\frac{y^2}{gs}}$, or more accurately $= \left(1 + \frac{h}{8r}\right) \pi \sqrt{\frac{y^2}{gs}}$. Inversely, the moment of inertia may be found from the time of oscillation of a suspended body, if we put:

$$My^2 = \left(\frac{t}{\pi}\right)^2 \cdot Mgs, \text{ or } y^2 = \left(\frac{t}{\pi}\right)^2 gs.$$

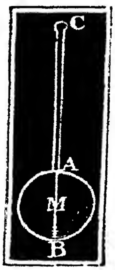
Fig. 306.



Examples.—1. For a uniform prismatic rod AB, Fig. 306, whose centre of oscillation is distant $CA = l_1$ and $CB = l_2$ from the extremities A and B, we have (§ 219) for the moment of inertia: $My^2 = \frac{1}{3} F (l_1^3 + l_2^3)$, and the statical moment $Ms = \frac{1}{2} F (l_1^2 - l_2^2)$; hence, the length of the mathematical pendulum which vibrates isochronously with this rod is $r = \frac{My^2}{Ms} = \frac{2}{3} \cdot \frac{l_1^3 + l_2^3}{l_1^2 - l_2^2} = \frac{l^2 + 3d^2}{6d}$, if l represent the sum $l_1 + l_2$, and d the difference $l_1 - l_2$. If this rod beats half seconds, we have $r = \frac{1}{4} \cdot \frac{g}{\pi^2} = \frac{1}{4} \cdot 0{,}10132 \cdot 32{,}2 = 0{,}8156$ feet $= 9{,}737$ inches, but if the entire length l of the rod amount to 12 inches, we must then put: $9{,}737 = \frac{144 + 3\,d^2}{6\,d} = d^2 - 19\,d = -48$ nearly; hence it follows:

$$d = \frac{19 - \sqrt{169}}{2} = 3 \text{ nearly; and from this}$$

Fig. 307.



$$l_1 = \frac{l+d}{2} = \frac{15}{2} = 7\tfrac{1}{2} \text{ inches, and } l_2 = \frac{l-d}{2} = \frac{9}{2} = 4\tfrac{1}{2} \text{ inches.}$$

—2. For a pendulum with a spherical lenticular bob AB, Fig. 307, if G be the weight and l the length CA of the rod or thread; K, on the other hand, the weight of the bob, and ρ its radius $MA = MB$:

$$r = \frac{\frac{1}{3} Gl^2 + K\left[(l+\rho)^2 + \frac{2}{5}\rho^2\right]}{\frac{1}{2} Gl + K(l+\rho)}.$$

If, now, the wire is 0,05 lbs., the bob 1,5 lbs., further, the length of the rod 1 foot, and the radius of the bob 1,15 inches, we then have the distance of the centre of oscillation of this pendulum from the axis of rotation:

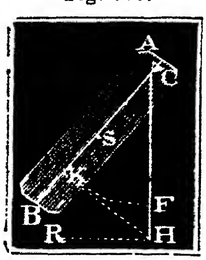
$$r = \frac{\frac{1}{3} \cdot 0{,}05 \cdot 12^2 + 1{,}5 \cdot (13{,}15^2 + \frac{2}{5} \cdot 1{,}15^2)}{\frac{1}{2} \cdot 0{,}05 \cdot 12 + 1{,}5 \cdot 13{,}15} = \frac{2{,}4 + 260{,}177}{0{,}3 + 19{,}725} = \frac{262{,}577}{20{,}025}$$

$= 13{,}112$ inches. Disregarding the rod, r would $= \frac{262{,}577}{19{,}725} = 13{,}312$ inches; and the inert mass of the bob being reduced to its centre, r would $= 13{,}15$ inches. The time of oscillation of this bob is:

$$t = \pi \sqrt{\frac{r}{g}} = 0{,}562 \sqrt{\frac{13{,}112}{12}} = 0{,}562 \sqrt{1{,}0926\ldots} = 0{,}5874 \text{ seconds.}$$

§ 251. The centre of suspension and centre of oscillation of a material pendulum are reciprocal, *i. e.* the one may be interchanged with the other, and the pendulum may be suspended at the centre of oscillation, without the time of oscillation being altered. The proof of this proposition may be given by aid of § 217, in the following manner. If T be the moment of inertia of the compound pendulum AB, Fig. 308, oscillating about the centre of gravity S, we have then for a revolution about the axis C, distant $CS = s$ from the centre of gravity S, $T_1 = T + Ms^2$, hence the distance of the centre of oscillation K from the axis of revolution C:

Fig. 308.

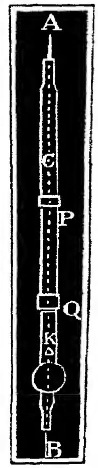


$$r = \frac{T_1}{MS} = \frac{T + Ms^2}{Ms} = \frac{T}{Ms} + s.$$

If now we represent the distance $KS = r - s$ of the centre of oscillation from the centre of gravity by s_1, we then obtain the simple equation $ss_1 = \frac{T}{M}$, in which s and s_1 appear in a similar manner, and hence may be substituted one for the other. This formula is not only

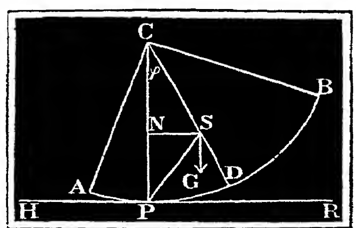
true for the descent, if s represents the distance of the centre of oscillation from the centre of gravity, but also inversely, if s expresses the distance of the centre of oscillation, and s_1 that of the centre of gyration from the centre of gravity, and C will therefore serve for the centre of oscillation if K serve for the centre of suspension. We avail ourselves of this property in the so-called *convertible pendulum* AB, Fig. 309, first proposed by Bohnenberger, and afterwards applied by Kater, which is furnished with two knife edges C and K, which are so situated with regard to each other, that the times of oscillation remain the same whether the pendulum oscillates about one or the other axis. In order that the axes may not be displaced with regard to each other, two sliding weights P and Q are applied, the smallest of which is attached by a fine screw. If by the shifting or adjustment of these weights, the time of oscillation comes to be the same, the pendulum may be suspended at C or at K, we shall then obtain in the distance CK of the two edges, the length r of the simple pendulum which vibrates synchronously with the convertible pendulum, and we shall now obtain the time of oscillation by the formula $t = \pi \sqrt{\frac{r}{g}}$.

Fig. 309.



§ 252. The swinging or rocking of a body with cylindrical base may be compared with the oscillations of a pendulum. This rocking, like every other rolling motion, is composed of a progressive and a rotary motion, but it may be assumed that it consists of a simple rotary motion with a variable axis of rotation. This axis of rotation is the point of support P, by which the vibrating body ABC, Fig. 310, rests on the horizontal base HR. If $CD = CP$ is the radius of the rolling base $ADB = r$, and the distance CS of the centre of gravity of the entire body from the centre C of this base $= s$, we have then for the distance corresponding to the angle of rotation $SCP = \phi$, $SP = y$ of the centre of gravity from the centre of gyration:

Fig. 310.



$$y^2 = r^2 + s^2 - 2\, r\, s\, cos.\, \phi = (r - s)^2 + 4\, r\, s \left(sin.\, \frac{\phi}{2}\right)^2;$$

hence, if further we represent the moment of inertia of the entire body about the centre of gravity S by Mk^2, we shall obtain the moment of inertia about the point of support P:

$$T = M\,(k^2 + y^2) = M\left[k^2 + (r - s)^2 + 4\, r\, s \left(sin.\, \frac{\phi}{2}\right)^2\right],$$

which for small angles of vibration may be put

$$= M\,[k^2 + (r - s)^2 + r\, s\, \phi^2],$$

or only $M[k^2 + (r - s)^2]$. Since now the moment of force $= G \,.\, SN = Mg \,.\, CS \sin.\phi = Mgs \sin.\phi$, it follows that the angular acceleration for the rotation about P:

$$\pi = \frac{\text{moment of force}}{\text{moment of inertia}} = \frac{Mgs \sin.\phi}{M[k^2+(r-s)^2]} = \frac{gs \sin.\phi}{k^2+(r-s)^2}.$$

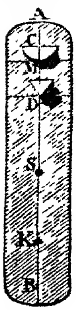
For the simple pendulum it is $= \frac{g \sin.\phi}{r_1}$, if r_1 represent is length; if both are to vibrate isochronously, it is necessary that:

$$\frac{gs \sin.\phi}{k^2+(r-s)^2} = \frac{g \sin.\phi}{r_1};\ i.\,e.\ r_1 = \frac{k^2+(r-s)^2}{s}.$$

The time of the vibration of the balance is from this:

$$t = \pi \sqrt{\frac{r_1}{g}} = \pi \sqrt{\frac{k^2+(r-s)^2}{gs}}.$$

Fig. 311.



This theory may also be applied to a pendulum AB, Fig. 311, with a rounded axis of rotation CM, if for r the radius of curvature CM of the axis be substituted. If, instead of the rounded axis, a knife edge D be applied, the time of vibration will then be

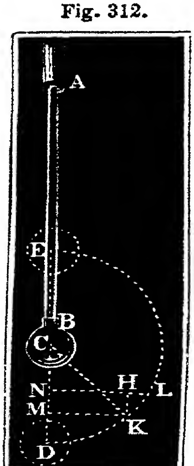
$$t_1 = \pi \sqrt{\frac{k^2 + \overline{DS}^2}{g\,DS}} = \pi \sqrt{\frac{k^2 + (s - x)^2}{g\,(s - x)}},$$

the distance CD of the edge from the centre of the round axis being represented by x. Both pendulums have equal times of vibration if

$$\frac{k^2+(s-x)^2}{s-x} = \frac{k^2+(r-s)^2}{s},\ \text{or}\ \frac{k^2}{s-x} - x = \frac{k^2+r^2}{s} - 2\,r.$$

If we write $\frac{k^2}{s-x} = \frac{k^2}{s} + \frac{k^2 x}{s^2}$ approximately, and neglect r^2, we shall then obtain $x = \frac{2\,r\,s^2}{s^2-k^2}$.

Fig. 312.



Remark 1. In the Second Part, under the article "Regulator," the conical pendulum will be mentioned.

Remark 2. *Elastic Pendulum.*—Bodies are likewise very often set into vibratory motion by elasticity. A string, or fine wire, AB, Fig. 312, is stretched by a weight $G = Mg$. If this weight is carried from the point of repose C to D, the string is thereby stretched $CD = r$; and if the weight be afterwards left to itself, it will, by virtue of the elasticity of the string, be raised again to C; it will arrive there with a certain velocity, and ascend by its *vis viva* to E, from whence it will again fall to D and C. In this manner the weight will oscillate a certain time in the space $DE = 2\ CD = 2\,r$ to and fro, and the question now is, as to its duration of oscillation. From the length $AB = l$, transverse section F and modulus of elasticity E of the string, it follows, § 183, that the force to extend it a length $CM = x$ is $P = \frac{x}{l} \,.\, FE$; hence, the mechanical effect required to extend it the length $\frac{r}{n}$ is $= \frac{Pr}{n} = \frac{x}{l} \,.\, \frac{r}{n} F\,E$. Let us now put successively $x = \frac{r}{n}, \frac{2r}{n}, \frac{3r}{n}$, &c., and add the corresponding mechanical effects, we shall then obtain the whole mechanical effect for the extension of the string.

$CD = r : L = \frac{r}{nl} \cdot FE \left(\frac{r}{n} + \frac{2r}{n} + \ldots\right) = \frac{r^2}{n^2 l} FE\,(1 + 2 + \ldots + n)$ $= \frac{r^2}{n^2 l} FE \cdot \frac{n^2}{2} = \frac{r^2}{2l} \cdot FE$; and for the extension $CM = x : L_1 = \frac{x^2}{2l} \cdot FE$. If now, inversely, the string be contracted by DM, therefore the weight D ascend from D to M, *i. e.* $r - x$, it will give the mechanical effect $L - L_1 = \left(\frac{r^2 - x^2}{2l}\right) FE$, and communicate to the weight G a velocity v corresponding to the *vis viva* $v^2 M = \frac{v^2}{g} \cdot G$; whence we shall have to put $\frac{v^2}{2g} G = \left(\frac{r^2 - x^2}{2l}\right) FE$, and the variable velocity of oscillation will be $v = \sqrt{\frac{FE}{Ml}} \sqrt{r^2 - x^2}$. But now $\sqrt{r^2 - x^2}$ may be put equal to the ordinate $MK = y$ of a semicircle described upon DE; hence it follows, more simply, that $v = \sqrt{\frac{FE}{Ml}} \cdot y$, and the instant for describing the particle of space $MN : \tau = \frac{MN}{y} \cdot \sqrt{\frac{Ml}{FE}}$. From the similarity of the triangles KLH and KCM, $\frac{KH}{KL} = \frac{KM}{KC}$, *i. e.* $\frac{MN}{KL} = \frac{y}{r}$, or $\frac{MN}{y} = \frac{KL}{r}$; hence, it follows that $\tau = \frac{KL}{r} \sqrt{\frac{Ml}{FE}}$; and lastly, the whole time of oscillation, or the time of describing the space DE is : $t = \frac{1}{r} \sqrt{\frac{Ml}{FE}}$ times the sum of all the elements of the semicircle $= \frac{1}{r} \sqrt{\frac{Ml}{FE}}$ times the semicircle

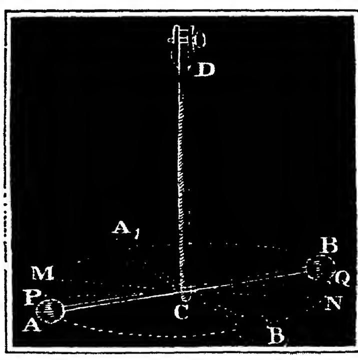
$$\pi r = \frac{\pi r}{r} \sqrt{\frac{Ml}{FE}} = \pi \sqrt{\frac{Ml}{FE}} = \frac{\pi}{\sqrt{g}} \sqrt{\frac{Gl}{FE}}.$$

If, for example, an iron wire, 20 feet long and 0,1 inch thick, be stretched by a weight $G = 100$ lbs. and set into longitudinal vibration, the duration of the oscillations will then be, since from § 186 $E = 26000000$,

$$t = \frac{\pi}{\sqrt{g}} \cdot \sqrt{\frac{100 \cdot 20}{(0{,}1)^2 \cdot \frac{\pi}{4} \cdot 26000000}}$$

$$= 0{,}553 \sqrt{\frac{2}{65 \cdot \pi}} = 0{,}05464 \text{ seconds.}$$

Fig. 313.



Remark 3. We have a torsion pendulum if a string or wire CD, Fig. 313, turns about an arm AB and is brought out of its natural position MN into the position AB, and then left to itself. The rod or arm AB is set into vibration by virtue of the torsion of the string, which extends to an equal distance on both sides of MN, so that $AM = A_1M$. If we put the force of torsion for the distance (1) and for the arc of vibration (1) $= K$, it will then be, for the angle of vibration $MCP = \phi^\circ$, $= K\phi$, and the corresponding mechanical effect $\frac{K\phi^2}{2}$; on the other hand, for the entire angle of elongation $MCA = \alpha = L_1 = \frac{K\alpha^2}{2}$. If now the inert mass of the entire pendulum $= M$ be reduced to the distance (r), and the angular velocity with which it passes from the position AB into that of $PQ = \omega$, we shall then have $\frac{M\omega^2}{2} = \frac{K(\alpha^2 - \phi^2)}{2}$; and, hence, $\omega = \sqrt{\frac{K}{M}(\alpha^2 - \phi^2)}$; and finally, the time of oscillation $t = \pi \sqrt{\frac{M}{K}}$.

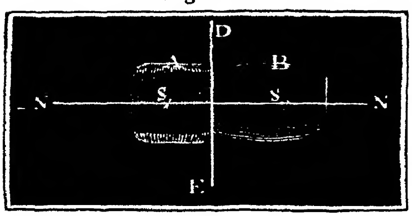
CHAPTER IV.

THE DOCTRINE OF IMPACT.

§ 253. *Impact in General.*—In virtue of the impenetrability of matter, two bodies cannot simultaneously occupy one and the same position. But when two bodies in motion come into contact with one another, so that the one strives to penetrate the space occupied by the other, a reciprocal action takes place, producing a consequent change in the conditions of motion of the two bodies. This reciprocal action is what is called *impact* or *collision.*

The relations of impact depend upon *the law of equality of action and reaction* (§ 62); during impact, the one body presses exactly as forcibly on the other as does this latter in an opposite direction on the former. The straight line, perpendicular to the surfaces in which the two bodies touch, and passing through the point of contact, is the direction of the impact. If the centres of gravity of the two bodies lie within this line, the impact is then called a *centric,* but if without, an *excentric, impact.* The bodies A and B, in Fig. 314, give a centric impact, because their centres of gravity S_1 and S_2 lie in the normal NN to the plane of contact DE; of the bodies A and B, Fig. 315, A thrusts centrically, and B excentrically, because S_1 lies within, and S_2 without, the normal line NN.

Fig. 314.



With respect to the direction of motion, we distinguish between *direct impact* and *oblique impact.* The direction of motion, in the case of direct impact, lies in the line of impact ; but in that of oblique impact, there is a deviation between the two directions. If, for example, the bodies A and B, Fig. 316, move in the directions $S_1\ C_1$

Fig. 315.

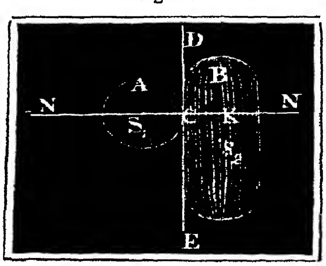
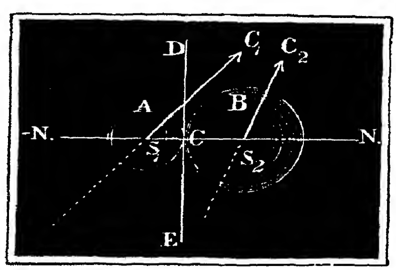


Fig. 316.



and S_2C_2, which deviate from the normal or line of impact NN, an oblique impact will take place; whilst, if the directions coincided with the normal, it would be direct.

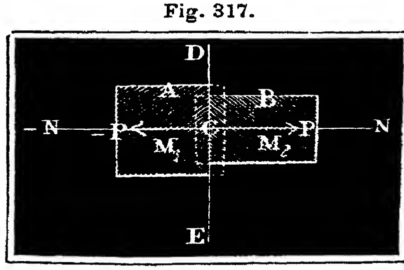
We make the further distinction of *the impact of free bodies* and *the impact of bodies entirely or partially supported.*

§ 254. The time occupied in the communication or change of motion by impact is indeed very small, but by no means indefinitely small; it depends, as well as the impact itself, upon the mass, velocity, and elasticity of the impinging bodies. We may regard this time as consisting of two periods. In the first period, the bodies become mutually compressed, and in the second, they again partially or entirely extend themselves. Elasticity is brought into action by this compression, and puts itself into equilibrium with the inertia, and thereby alters the state of motion of the impinging bodies. If the limit of elasticity is not exceeded by the compression, the body at the end of the impact perfectly recovers its former figure, and we then call it a *perfectly elastic* body; but if, at the end of the impact, a disfigurement takes place, we then call it an *imperfectly elastic body;* and lastly, if the body retains its original form, at a maximum pressure, and therefore has no tendency to expansion, we call it an *inelastic body.* At any rate, however, the distinction must only be taken as correct relatively to a certain strength of impact, for it is possible that one and the same body may show itself elastic to a weak, and inelastic to a stronger, impact. Strictly speaking, no body is perfectly elastic or perfectly inelastic; yet we shall, in the sequel, call bodies elastic which nearly recover their form after impact, and those inelastic which undergo a considerable and permanent disfigurement by impact (compare § 181).

In practical mechanics, impinging bodies, such as wood, iron, &c., are generally considered inelastic bodies, because they possess but little elasticity, and by repetition of the blows, lose still further that elasticity. It is a most important rule, moreover, to avoid, as far as possible, in machines and constructions, all jars or impacts, or so to moderate their effects as to convert them into elastic ones; because shocks and abrasions would be thereby produced, and a part of the mechanical effect consumed.

§ 255. *Inelastic Impact.*—Let us in the first place develop the laws of the direct central impact of freely moving bodies. Let us suppose the time of impact to be made up of equal parts τ, and let us assume that the pressure during the first instant is P_1, during the second P_2, during the third P_3, and so on. Let, now, the mass of the one body be $A = M$, Fig. 317, we shall then have the corresponding accelerating force:

Fig. 317.



$$p_1 = \frac{P_1}{M_1},\ p_2 = \frac{P_2}{M_1},\ p_3 = \frac{P_3}{M},\ \&c.;$$

but from § 19, the change of velocity due to the accelerating force p and particle of time τ is $\varkappa = p\,\tau$; hence, for the ensuing fall, we shall have the elementary increment or decrement: $\varkappa_1 = \frac{P_1\tau}{M_1}$, $\varkappa_2 = \frac{P_2\tau}{M_1}$, $\varkappa_3 = \frac{P_3\tau}{M_1}$, &c., and the consequent increment or decrement of velocity of the mass M_1 in a given finite time $\varkappa_1 + \varkappa_2 + \varkappa_3 + \ldots = (P_1 + P_2 + P_3 + \ldots)\frac{\tau}{M_1}$, as also the consequent change of velocity of the mass B of the magnitude M_2: $= (P_1 + P_2 + P_3 + \ldots)\frac{\tau}{M_2}$.

In the following or impinging body A, the pressure acts opposite to the velocity c_1, consequently here a decrement of velocity takes place; and after a certain time, the residuary velocity of the body is: $v_1 = c_1 - (P_1 + P_2 + \ldots)\frac{\tau}{M_1}$; in the preceding or impinged body B, on the other hand, the pressure acts in the direction of motion; hence there is an increment of velocity c_2, and it is converted into

$$v_2 = c_2 + (P_1 + P_2 + \ldots)\frac{\tau}{M_2}.$$

If we eliminate from both equations $(P_1 + P_2 + \ldots)\tau$, there will then remain the general formula:

I. $M_1(c_1 - v_1) = M_2(v_2 - c_2)$, or $M_1 v_1 + M_2 v_2 = M_1 c_1 + M_2 c_2$.

The product of the mass and velocity of a body is called the *momentum of the body*, and it may therefore be enunciated, *that for each instant of the time of impact, the aggregate of the momenta of the two bodies is as great as before impact.*

At the instant of maximum compression, both bodies have an equal velocity; hence, instead of v_1 and v_2, we may put this value into the equation found; then $M_1 v + M_2 v$ will remain $= M_1 c_1 + M_2 c_2$, and the velocity of the two bodies at the instant of maximum compression will be:

$$v = \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2}.$$

If the two bodies A and B are inelastic, they exert therefore no power after compression to re-expand themselves, and the communication of a change of motion will then cease, if both bodies are compressed to a maximum; and hence the two will go on after impact with a common velocity:

$$v = \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2}.$$

Examples.—1. An inelastic body B of 30 lbs. weight, moves with a 3 feet velocity, and is struck by another inelastic body A having a 7 feet velocity, the two will then proceed, after the blow, with the velocity

$v = \frac{50 \,.\, 7 + 30 \,.\, 3}{50 + 30} = \frac{350 + 90}{80} = \frac{44}{8} = \frac{11}{2} = 5\frac{1}{2}$ feet.—2. To cause a body of 120 lbs.

weight to pass from a velocity $c_2 = 1\frac{1}{2}$ feet into a 2 feet velocity v, it is struck by a body of 50 lbs. weight, what velocity will the body acquire? Here

$$c_1 = v + \frac{(v - c_2)\, M_2}{M_1} = 2 + \frac{(2 - 1{,}5) \,.\, 120}{50} = 2 + \frac{6}{5} = 3{,}2 \text{ feet.}$$

§ 256. *Elastic Impact.*—If the impinging bodies are perfectly elastic, they will then expand themselves after compression in the first period, gradually again in the second period of the time of impact, and when they have resumed the former shape, they will proceed in their motions with different velocities. But, since the mechanical effect which is expended on the compression of an elastic body is equal to the effect which the same gives out again by its expansion, no loss in *vis viva* will take place from the collision of elastic bodies, and hence the second following equation will be also true for this case:

$$\text{II. } M_1 v_1^2 + M_2 v_2^2 = M_1 c_1^2 + M_2 c_2^2, \text{ or}$$
$$M_1\,(c_1^2 - v_1^2) = M_2\,(v_2^2 - c_2^2).$$

From the equations I. and II., the velocities v_1 and v_2 of the bodies after impact may be found. First, it follows by division that $\frac{c_1^2 - v_1^2}{c_1 - v_1} = \frac{v_2^2 - c_2^2}{v_2 - c_2}$, i. e. $c_1 + v_1 = v_2 + c_2$, or $v_2 - v_1 = c_1 - c_2$; if now we put the resulting value of $v_2 = c_1 + v_1 - c_2$, into the equation I:

$$M_1 v_1 + M_2 v_1 + M_2\,(c_1 - c_2) = M_1 c_1 + M_2 c_2, \text{ or,}$$

$(M_1 + M_2)\, v_1 = (M_1 + M_2)\, c_1 - 2\, M_2\,(c_1 - c_2)$, from which we have the value:

$$v_1 = c_1 - \frac{2\, M_2}{M_1 + M_2}\,(c_1 - c_2), \text{ and}$$
$$v_2 = c_1 - c_2 + c_1 - \frac{2\, M_2\,(c_1 - c_2)}{M_1 + M_2} = c_2 - \frac{2\, M_1\,(c_1 - c_2)}{M_1 + M_2}.$$

Whilst for inelastic bodies the loss in velocity of the one body is

$$c_1 - v = c_1 - \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2} = \frac{M_2\,(c_1 - c_2)}{M_1 + M_2},$$

for elastic bodies it comes out twice as great, namely:

$$c_1 - v_1 = \frac{2 M_2\,(c_1 - c_2)}{M_1 + M_2},$$

and while the gain in velocity of the other body for inelastic bodies is:

$$v - c_2 = \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2} - c_2 = \frac{M_1\,(c_1 - c_2)}{M_1 + M_2},$$

for elastic bodies it is

$$v_2 - c_2 = \frac{2 M_1\,(c_1 - c_2)}{M_1 + M_2}, \text{ likewise twice as great.}$$

Example. Two perfectly elastic spheres, the one of 10 lbs. the other of 16 lbs. weight, impinge with the velocities 12 and 6 feet against each other, what will be their velocities after impact? Here $M_1 = 10$ and $c_1 = 12$ feet, but $M_2 = 16$ and $c_2 = -6$ feet, hence the loss of velocity of the first body will be

$$c_1 - v_1 = \frac{2 \,.\, 16\,(12 + 6)}{10 + 16} = \frac{2 \,.\, 16 \,.\, 18}{26} = 22{,}154 \text{ feet,}$$

and the gain in velocity of the other: $v_2 - c_2 = \frac{2 \,.\, 10 \,.\, 18}{26} = 13{,}846$ feet. From this the first body after impact will recoil with the velocity $v_1 = 12 - 22{,}154 = -10{,}154$ feet; and the other with that of $-6 + 13{,}846 = 7{,}846$ feet. Moreover, the measure of

vis viva of the two bodies after impact $= M_1v_1^2 + M_2v_2^2 = 10 \,.\, 10{,}154^2 + 16 \,.\, 7{,}846^2 = 1031 + 985 = 2016$, as likewise of that before impact, namely: $M_1c_1^2 + M_2c_2^2 = 10 \,.\, 12^2 + 16 \,.\, 6^2 = 1440 + 576 = 2016$. Were these bodies inelastic, the first would only lose in velocity $\frac{c_1 - v_1}{2} = 11{,}077$ feet, and the other gain $\frac{v_2 - c_2}{2} = 6{,}923$ feet; the first would still retain, after impact, the velocity $12 - 11{,}077 = 0{,}923$ feet, and the second acquire the velocity $- 6 + 6{,}923 = 0{,}923$, and the loss of mechanical effect would be $(2016 - (10 + 16)\, 0{,}923^2) \div 2g = (2016 - 22{,}2) \,.\, 0{,}0155 = 29{,}35$ ft. lbs.

§ 257. *Particular Cases.*—The formula developed in the foregoing paragraphs, for the final velocities of impact, hold good also in the case where the one body is at rest, or where both bodies move opposed to each other, or where the mass of the one is indefinitely great compared with the other. If the mass M_2 be at rest, we then have $c_2 = 0$, hence for the inelastic body

$$v = \frac{M_1c_1}{M_1 + M_2}, \text{ and for the elastic:}$$

$$v_1 = c_1 - \frac{2\,M_2c_1}{M_1 + M_2} = \frac{M_1 - M_2}{M_1 + M_2}\,c_1, \text{ and}$$

$$v_2 = 0 + \frac{2\,M_1c_1}{M_1 + M_2} = \frac{2\,M_1}{M_1 + M_2}\,c_1.$$

If the bodies meet, c_2 is therefore negative, and for an inelastic body it will follow that $v = \frac{M_1\,c_1 - M_2\,c_2}{M_1 + M_2}$, and for an elastic one:

$$v_1 = c_1 - \frac{2\,M_2\,(c_1 + c_2)}{M_1 + M_2}, \text{ and } v_2 = -\,c_2 + \frac{2\,M_1\,(c_1 + c_2)}{M_1 + M_2}.$$

If in this case the momenta be equal, $M_1c_1 = M_2c_2$, for the inelastic body then $v = 0$, *i. e.* the bodies bring each other to rest, but for elastic bodies:

$$v_1 = c_1 - \frac{2\,(M_2c_1 + M_1c_1)}{M_1 + M_2} = c_1 - 2\,c_1 = -\,c_1, \text{ and}$$

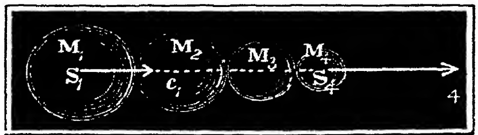
$$v_2 = -\,c_2 + \frac{2\,(M_2c_2 - M_1c_2)}{M_1 + M_2} = -\,c_2 + 2\,c_2 = +\,c_2;$$

then the bodies rebound after impact with opposite velocities. If on the other hand the masses are equal, we have then for inelastic bodies $v = \frac{c_1 - c_2}{2}$, and for elastic $v_1 = -\,c$, and $v_2 = c_1$, *i. e.* the masses rebound with their velocities interchanged.

If the masses again meet in the same direction, and if the preceding mass M_2 be indefinitely great, we shall then have for inelastic bodies $v = \frac{M_2c_2}{M_2} = c_2$, and for elastic $v_1 = c_1 - 2\,(c_1 - c_2) = 2\,c_2 - c_1$, $v_2 = c_2 + 0 = c_2$; the velocity therefore of the indefinitely great mass will not be altered by the collision of the finite mass. If, now, the indefinitely great mass be at rest, therefore, $c_2 = 0$, we shall then have for inelastic bodies $v = 0$, and for elastic $v_1 = -\,c_1$, $v_2 = 0$; the indefinitely great mass will then remain at rest, but in the first case, the impinging body will entirely lose its velocity, and in the second case this will be converted into an opposite one.

Examples.—1. With what velocity must a body of 8 lbs. impinge against another at rest of 25 lbs., in order that the last may have a velocity of 2 feet? Were the bodies inelastic, we should then have to put: $v = \frac{M_1 c_1}{M_1 + M_2}$, *i.e.* $2 = \frac{8 \cdot c_1}{8 + 25}$, hence $c_1 = \frac{33}{4} = 8\frac{1}{4}$ feet, the required velocity; but were they elastic, we should have $v_2 = \frac{2 M_1 c_1}{M_1 + M_2}$; hence, $c_1 = \frac{33}{8} = 4\frac{1}{8}$ feet.—2. If a sphere M_1, Fig. 318, strike against a mass at rest $M_2 = nM_1$ with the velocity c_1, the second, a third mass $M_3 = n\ M_2 = n^2\ M_1$ with the velocity communicated by the impact, this again another mass $M_4 = nM_3 = n^3\ M_1$, &c., we shall have from the perfect elasticity of these masses, the velocity

Fig. 318.



$$v_2 = \frac{2M_1}{M_1 + nM_1} c_1 = \frac{2}{1+n} \cdot c_1, v_3 = \frac{2\ M_2}{M_2 + n\ M_2}\ v_2 = \frac{2}{1+n} \cdot v_2 = \left(\frac{2}{1+n}\right)^2 c_1,$$

$v_4 = \left(\frac{2}{1+n}\right)^3 c_1$, &c. If, for example, the weight of each mass be half as great as that of the succeeding one, and we have therefore the exponents of the geometrical series formed by the masses: $n = \frac{1}{2}$, it will follow that

$$v_2 = \frac{4}{3} c_1, v_3 = \left(\frac{4}{3}\right)^2 c_1, v_4 = \left(\frac{4}{3}\right)^3 c_1 \ldots, v_{10} = \left(\frac{4}{3}\right)^9 c_1 = 13{,}32 \cdot c_1.$$

§ 258. *Loss of Mechanical Effect.*—In the collision of inelastic masses, a loss of *vis viva* constantly ensues, whence the masses after impact have not the power of producing so much mechanical effect, as before impact. Before impact the masses M_1 and M_2 proceeding with the velocities c_1 and c_2, contain the *vis viva*, $M_1 c_1^2 + M_2 c_2^2$, but after impact the masses proceeding with the velocity $v = \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2}$ have the *vis viva* $M_1 v^2 + M_2 v^2$; hence the subtraction of these forces will give the loss in *vis viva* by the collision: $K = M_1 (c_1^2 - v^2) + M_2 (c_2^2 - v^2) = M_1 (c_1 + v)(c_1 - v) - M_2 (c_2 + v)(v - c_2)$, but $M_1 (c_1 - v) = M_2 (v - c_2) = \frac{M_1 M_2 (c_1 - c_2)}{M_1 + M_2}$, hence

$$K = (c_1 + v - c_2 - v) \cdot \frac{M_1 M_2 (c_1 - c_2)}{M_1 + M_2} = \frac{(c_1 - c_2)^2 M_1 M_2}{M_1 + M_2} = \frac{(c_1 - c_2)^2}{\frac{1}{M_1} + \frac{1}{M_2}}.$$

If the weight of the masses are G_1 and G_2, M_1 is, therefore, $= \frac{G_1}{g}$, and $M_2 = \frac{G_2}{g}$, we shall from this have the loss in mechanical effect: $L = \frac{(c_1 - c_2)^2}{2g} \cdot \frac{G_1 G_2}{G_1 + G_2}$. We call $\frac{G_1 G_2}{G_1 + G_2}$ *the harmonic mean* of G_1 and G_2, and we may from this assert that *the loss in mechanical effect* which is produced by the impact of two inelastic masses, and which is expended upon the disfigurement of these, *is equivalent to the product of the harmonic mean of both masses, and of the height of fall which is due to the difference of the velocities of these masses.*

If one of the masses, for example M_2, be at rest, we shall have the

loss in mechanical effect $L = \frac{c_1^2}{2g} \cdot \frac{G_1 \, G_2}{G_1 + G_2}$, and if the mass moved M_1 be very great in comparison with the one at rest, G_1 will vanish as compared with G_2, and there will remain $L = \frac{c_1^2}{2g} \cdot G_2$.

For the rest we may put

$$K = M_1 (c_1^2 - v^2) + M_2 (c_2^2 - v^2) = M_1 (c_1^2 - 2\, c_1 v + v^2 + 2\, c_1 v - 2\, v^2) + M_2 (c_2^2 - 2\, c_2 v + v^2 + 2\, c_2 v - 2 v_2^2)$$
$$= M_1 (c_1 - v)^2 + 2\, M_1 v\, (c_1 - v) + M_2 (c_2 - v)^2 + 2\, M_2 v\, (c_2 - v)$$
$$= M_1 (c_1 - v)^2 + M_2 (c_2 - v)^2, \text{ because } M_1 (c_1 - v) = M_2 (v - c_2).$$

From this, therefore, *the vis viva lost by inelastic impacts is equivalent to the sum of the products of the masses and the squares of their loss or gain in velocity.*

Examples.—1. If in a machine, 16 blows per minute take place between two inelastic bodies $M_1 = \frac{1000}{g}$ lbs. and $M_2 = \frac{1200}{g}$ lbs., with the velocities $c_1 = 5$ feet, and $c_2 = 2$ feet, then the loss in mechanical effect from these blows will be:

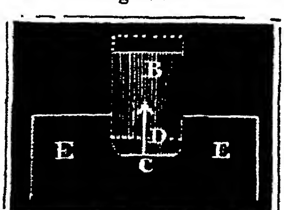
$$L = \frac{16}{60} \cdot \frac{(5-2)^2}{2g} \cdot \frac{1000 \, . \, 1200}{2200} = \frac{4}{15} \, . \, 9 \, . \, 0{,}0155 \, \frac{6000}{11} = 0{,}558 \, . \, \frac{400}{11} = 20{,}29$$

ft. lbs. per second.—2. If two trains upon a railroad of 120000 lbs. and 160000 lbs. weight, come into collision with the velocities $c_1 = 20$, and $c_2 = 15$ feet, there will ensue a loss of mechanical effect expended upon the destruction of the locomotives and carriages, which, in the case of perfect inelasticity of the impinging parts, will amount to

$$= \frac{(20 + 15)^2}{2g} \cdot \frac{120000 \, . \, 160000}{280000} = 35^2 \, . \, 0{,}0155 \, \frac{1920000}{28} = 1302000 \text{ ft. lbs.}$$

§ 259. *Pile Driving.*—The effects of impact are very often applied to ram or drive one body B, Fig. 319, into another E, a soft mass, for instance. If the resistance which the latter mass opposes to the penetration of the former be constant and $= P$, and the depth of penetration by one blow $= s$, a mechanical effect Ps will be then expended. If, on the other hand, this resistance at the commencement be $= 0$, and if it increase simultaneously with the depth of penetration, so that at the end, after the body has penetrated the second mass a depth s, it be $= P$, the mechanical effect expended will be then only $\frac{(0+P)}{2} s = \frac{1}{2} Ps$. If, lastly, the initial resistance be $= P_1$, and increase simultaneously with the space, so that, after describing a space s, it becomes P_2, we shall then have to put the mechanical effect $= \frac{(P_1 + P_2)}{2}$.

Fig. 319.



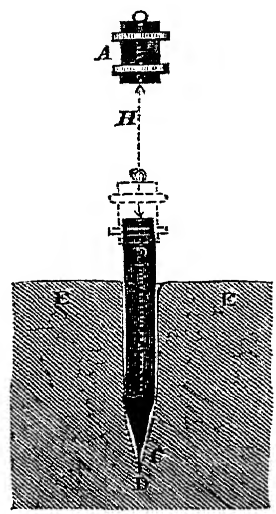
If the body B, whose mass may be M, begins with the velocity v to penetrate a mass, and if this velocity of penetration increase, it will, in virtue of its *vis viva*, have produced the mechanical effect $\frac{Mv^2}{2} = \frac{v^2}{2g} G$, if $G = Mg$ represent its weight.

When the resistance is constant, we must put: $Ps = \frac{v^2}{2g} G$; on the other hand, when the resistance beginning from nought gradually increases: $Ps = \frac{v^2}{2g} \cdot 2\,G$. ; and when it increases gradually from P_1 to P_2: $(P_1 + P_2)\, s = \frac{v^2}{2g} \cdot 2\,G$.

The initial velocity v is generated if a third mass A, whose magnitude may be $= M_1$ and weight $= G_1$, be allowed to impinge upon the second mass B, with a certain velocity c. If, now, these masses are inelastic, we then have the velocity with which the two proceed after impact, and begin to penetrate the mass E:

$$v = \frac{M_1 c}{M + M_1} = \frac{G_1 c}{G + G_1}.$$

Fig. 320.



In the driving of a *pile* or *post*, Fig. 320, B consists of a pile shod with iron, and A of a heavy body falling from a certain height, which is called a *ram*, or block of iron. If the height of fall $= H$, we shall have:

$$\frac{v^2}{2g} = \left(\frac{G_1}{G + G_1}\right)^2 \cdot \frac{c^2}{2g} = \left(\frac{G_1}{G + G}\right)^2 \cdot H,$$

hence the mechanical effect of the pile due to the velocity v

$$= \left(\frac{G_1}{G + G_1}\right)^2 GH,$$

and that of the pile and ram together

$$= \left(\frac{G_1}{G + G_1}\right)^2 (G + G_1)\, H = \frac{G_1^2\, H}{G + G_1}.$$

But if the resistance of the bed of earth be constant, the mechanical effect expended in the penetration of the pile will be $= Ps$, hence we shall have to put:

$$1.\ Ps = \left(\frac{G_1}{G + G_1}\right)^2 GH; \text{ or, } 2.\ Ps = \frac{G_1^2\, H}{G + G_1},$$

the first if the ram does not, during penetration, remain upon the pile, and the second if both go down together.

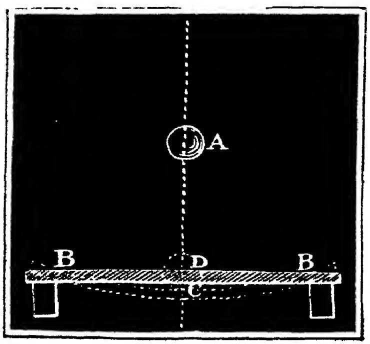
The weight $G + G_1$ produces, in penetrating, the mechanical effect $(G + G_1)\, s$, we may then more correctly put: $(P - G - G_1)\, s = \frac{G_1^2\, H}{G + G_1}$; but $G + G_1$ is small compared with P, and may generally be neglected.

Hence, were the impact perfectly elastic, we should have to put: $Ps = \left(\frac{2\,G_1}{G + G_1}\right)^2 \cdot GH$. Were G small, compared with G_1, as for instance, in the driving of a nail, we should have either $Ps = G_1 H$, or $Ps = 4\,GH$.

Example. A pile of 400 lbs. weight is driven by the last round of 20 blows of a 700 lbs. ram, falling from a height of 5 feet, 6 inches deeper, what resistance will the ground offer, or what load will the pile sustain without penetrating deeper? Here $G = 400$, $G_1 = 700$ lbs., $H = 5$, and $s = \frac{0,5}{20} = 0,025$ feet, whereby it is supposed that the pile penetrates equally far for each blow. From the first formula $P = \left(\frac{700}{700 + 400}\right)^2 \frac{400 \,.\, 5}{0,025} = \left(\frac{7}{11}\right)^2 . \, 80000 = 32400$ lbs.; and from the second: $P = \frac{700^2 \,.\, 5}{1100 \,.\, 0,025} = \frac{4900}{11} . \, 200 = 89100$ lbs.

For duration, with security, such piles are only loaded from $\frac{1}{100}$ to $\frac{1}{10}$ of their strength.

Fig. 321.



§ 260. The formulæ found above are applicable to the breaking of bodies by descending weights or balls. Let BB, Fig. 321, be a prismatic body of the mass M, or weight $G = Mg$, supported at its extremities, which is bent a depth $CD = s$, by a weight G_1 falling from a height $AD = H$ upon its middle, and in this manner broken; the conditions under which this is possible are to be determined.

From § 190, the deflexion, or the height of the arc, is given $s = \frac{P l^3}{48 \, WE}$, from the pressure P in the middle of the beam, and from its length $BB = l$, and if further its moment of flexure WE is known; therefore, inversely, the pressure corresponding to a certain deflexion s is: $P = \frac{48 \, WEs}{l^3}$. This pressure however is not constant, but increases simultaneously with s, hence the mechanical effect expended in the deflexion by a depth s, not $= Ps$, but only $\frac{1}{2} Ps$, *i. e.* $\frac{1}{2} . \frac{P^2 \, l^3}{48 \, WE} = \frac{P^2 \, l^3}{96 \, WE}$. This mechanical effect may now be equated to that which the falling body communicates to the beam. Since the beam rests on its extremities, we must (from § 219) consider only the third part of its mass as inert, and hence put for this mechanical effect:

$$\left(\frac{G_1}{\frac{1}{3} G + G_1}\right)^2 . \, \frac{1}{3} GH, \text{ or } \frac{G_1^2 H}{\frac{1}{3} G + G_1}.$$

The first, if the weight G_1 flies back after its descent, and the second if it remains on the beam during the fracture.

If we suppose a rectangular beam of the depth h, and breadth b, we shall then have to put: $W = \frac{1}{12} bh^3$, and $P = \frac{4}{6} \frac{bh^2}{l} K$, hence $\frac{P^2}{W} = \frac{16}{3} . \frac{bhK^2}{l^2}$; accordingly, the mechanical effect for the rupture

of the beam will be $= \frac{1}{18} \cdot \frac{P^2 l^3}{WE} = \frac{bhlK^2}{18\,E}$, and we may now put:

$$1.\ GH\left(\frac{G_1}{\frac{1}{3}\,G + G_1}\right)^2 = \frac{bhlK^2}{6\,E}, \text{ or } 2.\ \frac{G_1{}^2 H}{\frac{1}{3}\,G + G_1} = \frac{bhlK^2}{18\,E}.$$

Example.—From what height must an iron weight G_1 of 100 lbs. be allowed to fall to break a cast-iron plate, 36 inches long, 12 inches broad, and 3 inches thick, in its middle? The modulus of elasticity of cast-iron $E = 17000000$, and the modulus of strength $K = 19000$, hence it follows that:

$$\frac{bhlK^2}{6\,E} = \frac{12\,.\,3\,.\,36\,.\,19000^2}{6\,.\,17000000} = \frac{216\,.\,19^2}{17} = \frac{216\,.\,361}{17} = 4587.$$

If now a cubic inch of cast iron weighs 0.275* lbs., the weight of a plate G will then be $= 12\,.\,3\,.\,36\,.\,0{,}275 = 1296\,.\,0{,}275 = 356{,}4$ lbs.; hence:

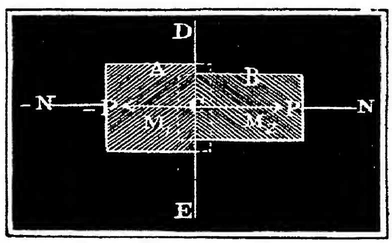
$$G\left(\frac{G_1}{\frac{1}{3}\,G + G_1}\right)^2 = 356{,}4\,.\left(\frac{100}{218{,}8}\right)^2 = 74{,}44; \text{ on the other hand,}$$

$$\frac{G_1{}^2}{\frac{1}{3}\,G + G_1} = \frac{10000}{218{,}8} = 45{,}70. \text{ Hence the height of fall required is:}$$

$$H = \frac{4587}{74{,}44} = 61{,}6 \text{ inches, or } H = \frac{4587}{3\,.\,45{,}7} = 33{,}5 \text{ inches.}$$

§ 261. *Hardness.*—When the modulus of elasticity of the impinging bodies is known, we may then find the force of compression and its amount. Let the transverse sections of the bodies A and B, Fig. 322, be F_1 and F_2, the lengths l_1 and l_2, and the moduli of elasticity E_1 and E_2. If both impinge against each other with a force P, the compressions effected will be from § 183:

Fig. 322.



$$\lambda_1 = \frac{Pl_1}{F_1E_1}, \text{ and } \lambda_2 = \frac{Pl_2}{F_2E_2},$$

and their ratio:

$$\frac{\lambda_1}{\lambda_2} = \frac{F_2E_2}{F_1E_1} \cdot \frac{l_1}{l_2}.$$

If for simplicity we represent $\frac{FE}{l}$ by H, we obtain $\lambda_1 = \frac{P}{H_1}$, and $\lambda_2 = \frac{P}{H_2}$, as well as $\frac{\lambda_1}{\lambda_2} = \frac{H_2}{H_1}$. If, after the example of Whewell,† we call the quantity $\frac{FE}{l}$ the hardness of a body, it follows that *the depth of compression is inversely proportional to the hardness.*

If a mass $M = \frac{G}{g}$ impinges with the velocity c upon an immovable or indefinitely great mass, it then expends its whole *vis viva* upon the compression, hence $\frac{1}{2}\,Ps = \frac{Mc^2}{2} = \frac{c^2}{2g}\,G$. But now the space s is equal to the aggregate of the compressions λ_1 and λ_2, and λ_1

* A nearer statement of its weight is, 0,2604 lbs. avoirdupois.—Am. Ed.
† The Mechanics of Engineering, § 207.

$= \frac{P}{H_1}$, and $\lambda_2 = \frac{P}{H_2}$; hence it follows that

$$s = \lambda_1 + \lambda_2 = P\left(\frac{1}{H_1} + \frac{1}{H_2}\right) = \frac{H_1 + H_2}{H_1 H_2} \cdot P,$$

as, inversely, $P = \frac{H_1 H_2}{H_1 + H_2}\, s$, and the equation of condition $\frac{1}{2}$ · $\frac{H_1 H_2}{H_1 + H_2} \cdot s^2 = \frac{c^2}{2g}\, G$, therefore,

$$s = c\sqrt{\frac{H_1 + H_2}{H_1 H_2} \cdot \frac{G}{g}}.$$

from which P, λ_1 and λ_2 may be calculated.

Example. If a wrought-iron hammer, of 4 square inches base and 6 inches high, strikes with a velocity of 50 feet upon a plate of lead, of 2 square inches base and 1 inch thick, the following relations present themselves. The modulus of elasticity of wrought iron is $E_1 = 29000000$, and that of lead $E_2 = 700000$; hence, the hardness of these bodies is: $H_1 = \frac{F_1 E_1}{l_1} = \frac{4 \cdot 29000000}{6} = 19333333$, and $H_2 = \frac{F_2 E_2}{l_2} = \frac{2 \cdot 700000}{1} = 1400000$. If we put these values into the formula $s = c\sqrt{\frac{H_1 + H_2}{H_1 H_2} \cdot \frac{G}{g}}$, and substitute for the weight of the hammer $= 4 \cdot 6 \cdot 0{,}29 = 7$ lbs.; therefore, $\frac{G}{g} = 7 \cdot 0{,}031 = 0{,}217$, we shall then obtain the space of the hammer in the compression:
$s = 50\sqrt{\frac{20733333 \cdot 0{,}224}{19333333 \cdot 1400000}} = 50\sqrt{\frac{0{,}46443}{2706666}} = 0{,}0207$ inches $= 0{,}249$ lines. From this the force of impact or pressure follows:
$P = \frac{H_1 H_2}{H_1 + H_2} \cdot s = \frac{19333333 \cdot 1400000}{20733333} \cdot 0{,}0207 = 27037$ lbs.; further, the compression of the hammer is: $\lambda_1 = \frac{P}{H_1} = \frac{27037}{19333333} = 0{,}0014$ inches $= 0{,}016$ lines, and that of the leaden plate: $\lambda_2 = \frac{P}{H_2} = \frac{27037}{1400000} = 0{,}0193$ inches $= 0{,}233$ lines.

§ 262. *Elastic and Inelastic Impact.*—If two masses M_1 and M_2 move with the velocities c_1 and c_2, the common velocity of the two at the moment of maximum compression will then be from § 256 $v = \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2}$, and the mechanical effect expended on the compression from § 259:

$$L = \frac{(c_1 - c_2)^2}{2} \cdot \frac{M_1 M_2}{M_1 + M_2} = \frac{(c_1 - c_2)^2}{2g} \cdot \frac{G_1 G_2}{G_1 + G_2}.$$

This mechanical effect may be also put:

$$= \tfrac{1}{2} Ps = \tfrac{1}{2} P(\lambda_1 + \lambda_2) = \tfrac{1}{2} \cdot \frac{H_1 H_2}{H_1 + H_2}\, s^2,$$

consequently the sum of the compressions of both masses will be:

$$s = (c_1 - c_2)\sqrt{\frac{G_1 G_2}{g(G_1 + G_2)} \cdot \frac{H_1 + H_2}{H_1 H_2}},$$

from which the compressing force P, and the compressions of the separate masses λ_1 and λ_2, may be found.

If the masses are inelastic, these compressions will remain after impact, but if only one of the two bodies be inelastic, the other will

again recover its form in the second period, and produce a mechanical effect which will generate a new change of velocity. If, for example, $M_1 = \frac{G_1}{g}$ be elastic, the mechanical effect in this second period of impact $\frac{1}{2} P\lambda_1 = \frac{1}{2} \cdot \frac{P^2}{H_1} = \frac{1}{2 H_1}\left(\frac{H_1 H_2}{H_1 + H_2}\right)^2 s^2 = \frac{(c_1 - c_2)^2}{2g} \cdot \frac{G_1 G_2}{G_1 + G_2} \cdot \frac{H_2}{H_1 + H_2}$ will be given out; hence we shall have in this case for the velocities v_1 and v_2 after impact the formula:

$$M_1 v_1 + M_2 v_2 = M_1 c_1 + M_2 c_2, \text{ and}$$

$$M_1 v_1^2 + M_2 v_2^2 = M_1 v^2 + M_2 v^2 + (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{H_2}{H_1 + H_2}$$

$$= M_1 c_1^2 + M_2 c_2^2 - (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} + (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{H_2}{H_1 + H_2},$$

i. e. $M_1 v_1^2 + M_2 v_2^2 = M_1 c_1^2 + M_2 c_2^2 - (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{H_1}{H_1 + H_2}$.

If the loss of velocity $c_1 - v_1$ be put $= x$, we shall then have the gain of velocity $v_2 - c_2 = \frac{M_1 x}{M_2}$, and the last equation will assume the form:

$$x\,(2\,c_1 - x) - x\left(2\,c_2 + \frac{M_1 x}{M_2}\right) - (c_1 - c_2)^2 \frac{M_2}{M_1 + M_2} \cdot \frac{H_1}{H_1 + H_2} = 0, \text{ or,}$$

$$\frac{M_1 + M_2}{M_2}\, x^2 - 2\,(c_1 - c_2)\, x + (c_1 - c_2)^2 \cdot \frac{M_2}{M_1 + M_2} \cdot \frac{H_1}{H_1 + H_2} = 0.$$

If this be multiplied by $\frac{M_2}{M_1 + M_2}$ and $\frac{H_1}{H_1 + H_2}$ be put $= 1 - \frac{H_2}{H_1 + H_2}$, we shall then obtain the quadratic equation:

$$x^2 - 2\,(c_1 - c_2)\,\frac{M_2}{M_1 + M_2}\, x + (c_1 - c_2)^2 \left(\frac{M_2}{M_1 + M_2}\right)^2$$

$$= (c_1 - c_2)^2 \left(\frac{M_2}{M_1 + M_2}\right)^2 \cdot \frac{H_2}{H_1 + H_2}, \text{ or,}$$

$$\left(x - (c_1 - c_2)\frac{M_2}{M_1 + M_2}\right)^2 = (c_1 - c_2)^2 \left(\frac{M_2}{M_1 + M_2}\right)^2 \cdot \frac{H_2}{H_1 + H_2},$$

whose solution will give x, or the loss in velocity of the first body:

$$c_1 - v_1 = (c_1 - c_2)\,\frac{M_2}{M_1 + M_2}\left(1 + \sqrt{\frac{H_2}{H_1 + H_2}}\right),$$

and the gain in velocity of the second:

$$(v_2 - c_2 = c_1 - c_2)\,\frac{M_1}{M_1 + M_2}\left(1 + \sqrt{\frac{H_2}{H_1 + H_2}}\right).$$

Example. If we assume the iron hammer in the example of the preceding paragraph to be perfectly elastic and the plate of lead inelastic, we shall then obtain the loss in velocity of the 7 lbs. hammer, descending with a 50 feet velocity, since $c_2 = 0$ and $M_2 = \infty$.

$$c_1 - v_1 = c_1\left(1 + \sqrt{\frac{H_2}{H_1 + H_2}}\right) = 50\left(1 + \sqrt{\frac{1400000}{20733333}}\right)$$

$= 50\,(1 + 0{,}26) = 63$ feet;
hence, the velocity of the hammer after the blow is : $v_1 = c_1 - 63 = 50 - 63 = -13$ feet. The velocity of the supported plate of lead $= 0$.

§ 263. *Imperfectly Elastic Impact.*—If the bodies impinging against each other are imperfectly elastic, they only partially recover their figure in the second period of the time of impact, the *vis viva* expended in compression during the first period, will not, therefore, again be completely given out. If, again, λ_1 and λ_2 are the depths of penetration, and P the pressure, we shall then have the loss of mechanical effect by the compression $= \frac{1}{2} P\lambda_1$, and $\frac{1}{2} P\lambda_2$, and if during the expansion the μth part of this, or generally during the expansion of the one body the μ_1th, and during that of the second the μ_2th be given back, there will remain the aggregate loss of mechanical effect after impact:

$$L = \tfrac{1}{2} P\,[(1 - \mu_1)\,\lambda_1 + (1 - \mu_2)\,\lambda_2], \text{ or } \lambda_1 = \frac{P}{H_1}, \text{ and } \lambda_2 = \frac{P}{H_2},$$

$$L = \tfrac{1}{2} P^2 \left[\frac{1 - \mu_1}{H_1} + \frac{1 - \mu_2}{H}\right].$$ But from the former paragraph:

$$P = \frac{H_1 H_2 s}{H_1 + H_2}, \text{ and } s = (c_1 - c_2) \sqrt{\frac{M_1 M_2}{M_1 + M_2} \cdot \frac{H_1 + H_2}{H_1 H_2}},$$

hence the loss of the mechanical effect in question is known:

$$L = \frac{(c_1 - c_2)^2}{2} \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{H_1 H_2}{H_1 + H_2} \left(\frac{1 - \mu_1}{H_1} + \frac{1 - \mu_2}{H_2}\right).$$

Now, in order to find the velocities v_1 and v_2 after impact, we have to combine them with each other and to solve the equations:

$$M_1 v_1 + M_2 v_2 = M_1 c_1 + M_2 c_2, \text{ and}$$
$$M_1 v_1^2 + M_2 v_2^2 = M_1 c_1^2 + M_2 c_2^2 - (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{(1 - \mu_1)\,H_2 + (1 - \mu_2)\,H_1}{H_1 + H_2}.$$

In the same manner as in the former §, the *loss of velocity of the first body is given:*

$$c_1 - v_1 = (c_1 - c_2)\,\frac{M_2}{M_1 + M_2}\left(1 + \sqrt{\frac{\mu_2 H_1 + \mu_1 H_2}{H_1 + H_2}}\right),$$

and the *gain of velocity of the body preceding:*

$$v_2 - c_2 = (c_1 - c_2)\,\frac{M_1}{M_1 + M_2}\left(1 + \sqrt{\frac{\mu_2 H_1 + \mu_1 H_2}{H_1 + H_2}}\right).$$

These two general formulæ also embrace the laws of perfectly elastic and perfectly inelastic impact. If in them we put $\mu_1 = \mu_2 = 1$, we then obtain the formula already found above for the impact of perfectly elastic bodies, but if we assume $\mu_1 = \mu_2 = 0$, we then obtain the formula for inelastic impact, &c. If both bodies have the same degree of elasticity, therefore, $\mu_1 = \mu_2$, we have more simply:

$$c_1 - v_1 = (c_1 - c_2)\,\frac{M_2}{M_1 + M_2}\,(1 + \sqrt{\mu}), \text{ and}$$

$$v_2 - c_2 = (c_1 - c_2)\,\frac{M_1}{M_1 + M_2}\,(1 + \sqrt{\mu}).$$

If, further, the mass M_2 is at rest, and infinitely great, it then follows that:

$$c_1 - v_1 = c_1 (1 + \sqrt{\mu}), \text{ i. e. } v_1 = - c_1 \sqrt{\mu},$$

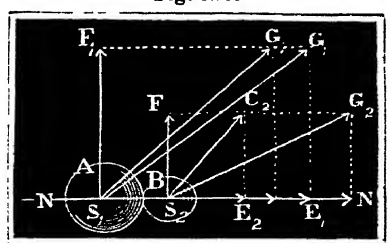
as inversely, $\mu = \left(\frac{v_1}{c_1}\right)^2$. If now M_1 be allowed to fall from a height h upon a similar mass M_2, and if it reascend to a height h_1, we may then find from the two the co-efficient of imperfect elasticity, by the formula $\mu = \frac{h_1}{h}$. Newton has already found in this manner for ivory $\mu = \left(\frac{8}{9}\right)^2 = \frac{64}{81} = 0{,}79$; for glass $\mu = \left(\frac{15}{16}\right)^2 = 0{,}9375^2 = 0{,}879$; for cork, steel, and wool, $\mu = \left(\frac{5}{9}\right)^2 = 0{,}555^2 = 0{,}309$. It must be here supposed that the impinging, or striking body, is spherical, and the body impinged upon, or the support, flat.

Example. What velocities will two steel plates acquire after impact, if they possessed before impact the velocities $c_1 = 10$ and $c_2 = -6$ feet, the one weighs 30 the other 40 lbs.? Here

$c_1 - v_1 = (10 + 6) \frac{40}{70} \left(1 + \frac{5}{9}\right) = 16 \cdot \frac{4}{7} \cdot \frac{14}{9} = \frac{16 \cdot 8}{9} = 14{,}22$ feet; hence, the velocities sought are $v_1 = c_1 - 14{,}22 = 10 - 14{,}22 = -4{,}22$ feet, and $v_2 = c_2 + 10{,}66 = -6 + 10{,}66 = 4{,}66$ feet.

§ 264. *Oblique Impact.*—If the directions of motion $\overline{S_1C_1}$ and $\overline{S_2C_2}$ of two bodies A and B, Fig. 323, deviate from the normal $\overline{NN}$ to the plane of contact, the impact is then *oblique*. We may reduce the theory of this to that of direct impact if we resolve the velocities $S_1C_1 = c_1$, and $S_2C_2 = c_2$, in a normal and tangential direction; the lateral velocities in the direction of the normal $\overline{NN}$ communicate a certain impact, and hence are altered to the same amount as for centric impact; the velocities, on the other hand, parallel to the plane of contact, communicate no impact, and hence remain unaltered. If we join the normal velocity of a body changed in accordance with the laws of centric impact to the remaining unchanged tangential velocity, we shall obtain the resultant velocities of these bodies after impact. If we represent the angles which the directions of motion make with the normal by α_1 and α_2, then $C_1S_1N = \alpha_1$, and $C_2S_2N = \alpha_2$, we shall obtain for the normal velocities S_1E_1 and S_2E_2 the values $c_1 \cos. \alpha_1$ and $c_2 \cos. \alpha_2$, for the tangential velocities on the other hand S_1F_1 and S_2F_2: $c_1 \sin. \alpha_1$ and $c_2 \sin. \alpha_2$. The first velocities suffer alteration from the effect of the impact, and the one passes into

Fig. 323.



$$v_1 = c_1 \cos. \alpha_1 - (c_1 \cos. \alpha_1 - c_2 \cos. \alpha_2) \frac{M_2}{M_1 + M_2} (1 + \sqrt{\mu}),$$

and the second into:

$$v_2 = c_2 \cos. \alpha_2 + (c_1 \cos. \alpha_1 - c_2 \cos. \alpha_2) \frac{M_1}{M_1 + M_2}(1 + \sqrt{\mu}),$$

M_1 and M_2 representing the masses of the bodies.

The resultant velocity $S_1 G_1$ of the first body is given by v_1 and $c_1 \sin. \alpha_1$, $w_1 = \sqrt{v_1^2 + c_1^2 \sin. \alpha_1^2}$, and the velocity $S_2 G_2$ of the second body by v_2 and $c_2 \sin. \alpha_2$; $w_2 = \sqrt{v_2^2 + c_2^2 \sin. \alpha_2^2}$; the deviations from the normal are also given by the formula:

$$tang. \phi_1 = \frac{c_1 \sin. \alpha_1}{v_1}, \text{ and } tang. \phi_2 = \frac{c_2 \sin. \alpha_2}{v_2},$$

ϕ_1 representing the angle $G_1 S_1 N$ and ϕ_2 the angle $G_2 S_2 N$.

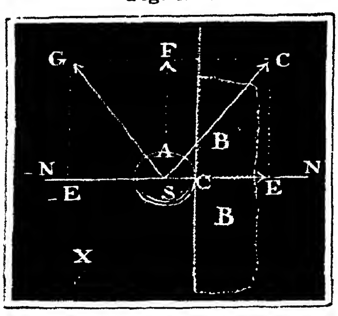
Example. Two spheres of 30 and 50 lbs. impinge against each other with the velocities $c_1 = 20$ and $c_2 = 25$ feet, which deviate from the normal by the angle $\alpha_1 = 21° 35'$ and $\alpha_2 = 65° 20'$, in what directions and with what velocities will the two bodies proceed after impact? The uniform component velocities are: $c_1 \sin. \alpha_1 = 20 . \sin. 21° 35' = 7.357$ feet, and $c_2 \sin. \alpha_2 = 25 . \sin. 65° 20' = 22,719$ feet; the variable, on the other hand, $c_1 \cos. \alpha_1 = 20 . \cos. 21° 35' = 18,598$ feet, and $c_2 \cos. \alpha_2 = 25 . \cos. 65° 20', = 10,433$ feet. If the bodies are inelastic, then $\mu = 0$; hence, the altered normal velocities are:

$v_1 = 18,598 - (18,598 - 10,433) \frac{50}{80} = 18,598 - 5,103 = 13,495$ feet, and $v_2 = 10,433 + 8,165 . \frac{3}{8} = 10,433 + 3,062 = 13,495$ feet. The resultant velocities are now:

$$w_1 = \sqrt{13,495^2 + 7,357^2} = \sqrt{236,24} = 15,37 \text{ feet, and}$$
$$w_2 = \sqrt{13,495^2 + 22,719^2} = \sqrt{698,27} = 26,42 \text{ feet; and}$$

we have for their directions the $tang. \phi_1 = \frac{7,357}{13,495}$, $log. tang. \phi_1 = 0,73653 - 1$, $\phi_1 = 28° 36'$ and $tang. \phi_2 = \frac{22,719}{13,495}$, $log. tang. \phi_2 = 0,22622$, $\phi_2 = 59° 17'$.

Fig. 324.



§ 266. If a mass A, Fig. 324, strikes against another mass, indefinitely great, or against an immovable resistance BB, we have $c_2 = 0$ and $M_2 = \infty$, it then follows that

$$v_1 = c_1 \cos. \alpha_1 - c_1 \cos. \alpha_1 (1 + \sqrt{\mu})$$
$$= - c_1 \cos. \alpha_1 \sqrt{\mu} \text{ and}$$
$$v_2 = 0 + c_1 \cos. \alpha_1 . \frac{M_1 (1 + \sqrt{\mu})}{\infty}$$
$$= 0 + 0 = 0;$$

if now, further, $\mu = 0$, v_1 will also $= 0$; but if $\mu = 1$, v_1 will $= - c_1 \cos. \alpha_1$; *i. e. in inelastic impact, the normal velocity is entirely lost; in elastic, on the other hand, it is changed in the opposite direction.* For the angle by which the direction of motion after impact deviates from the normal $tang. \phi_2$ is

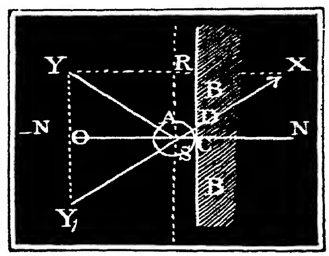
$$= \frac{c_1 \sin. \alpha_1}{v_1} = - \frac{c_1 \sin. \alpha_1}{c_1 \cos. \alpha_1 \sqrt{\mu}} = - tang. \alpha_1 \sqrt{\frac{1}{\mu}};$$

for inelastic bodies the $tang. \phi_1$ is therefore $= - \frac{tang. \alpha_1}{0} = \infty$, *i. e.* $\phi_1 = 90°$, and for elastic $tang. \phi_1 = - tang. \alpha_1$, *i. e.* $\phi_1 = - \alpha_1$. After

the impact of an inelastic body against an inelastic resistance, the first proceeds with a tangential velocity c_1 *sin.* α_1 in the direction SF of the plane of contact; after the impact of an elastic body against an elastic resistance, the body proceeds with a uniform velocity in the direction SG, which lies in a plane with the normal $\overline{NN}$ and the initial direction XS, and makes with the normal the same angle $GS\overline{N}$ as does the direction of motion with it before impact, but on the opposite side. The angle $XS\overline{N}$ which the direction of motion before impact makes with the normal or the vertical, is called the *angle of incidence*, and the angle GSN, which the direction of motion after impact makes with the same, the *angle of reflexion;* and it may therefore be enunciated, *that in perfectly elastic impact, the angles of reflexion and incidence lie in the same plane with the vertical, and are equal to each other.*

In imperfectly elastic impact, the ratio $\sqrt{\mu}$ of the tangents of these angles, is equal to the ratio of the velocity given back by the expansion to that of the velocity lost by the compression. By the aid of this law, the direction may be easily found in which the body A, Fig. 325, must impinge against the immovable resistance BB, that after impact it may pursue a certain direction SY. If the impact be elastic we must let fall from a point Y of the given direction the perpendicular YO on the incident perpendicular $\overline{NN}$, prolong the same until the prolongation OY_1 is equal to the perpendicular: SY_1 is then the direction of the impact in question, for from this construction, the angle $NSY_1 = NSY$. If the impact be inelastic, we may make $OY_1 = \sqrt{\mu} \,.\, OY$, then Y_1S will be likewise the initial direction sought, since $\frac{tang.\ \alpha_1}{tang.\ \phi_1} = \frac{OY_1}{OY}$, and also $= \sqrt{\mu}$.

Fig. 325.



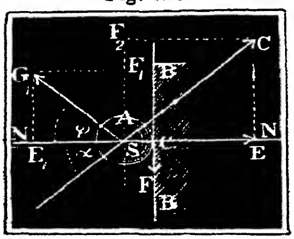
If a perpendicular YR be let fall upon the line SR parallel to the plane of contact, and its prolongation RX be made $= \sqrt{\frac{1}{\mu}}\ RY$, we shall then also obtain, for evident reasons, in SX the direction of the incidence sought.

Remark. The theory of oblique impact has especial application to the game of billiards. (*See* Théorie mathématique des effets du jeu de billard," par Coriolis.) According to Coriolis, in the striking of a billiard-ball against the cushion, the ratio of the reflected velocity to that of the incident = 0,5 to 0,6; therefore $\mu = 0{,}5^2 = 0{,}25$ to $0{,}6^2 = 0{,}36$. With the assistance of this value, the direction may now be known in which a ball A must strike against the cushion BB that it may be reflected from this towards a given point Y. We may let fall from the given point Y the perpendicular YR to the line of gravity of the ball parallel to the cushion, prolong it by $RX = \sqrt{\frac{1}{\mu}} = \frac{10}{6}$ to $\frac{10}{5}$ of its value, and draw the straight line Y_1X: the point of intersection D is the place where the

ball A must be struck that it may rebound to Y. By the motion of rotation of the ball, this ratio will be somewhat altered.

§ 267. In oblique impact a friction takes place between the impinging bodies, which changes the lateral velocities in the direction of the plane of contact. The friction of impact is determined like that of the friction of pressure, P representing the pressure, and f the co-efficient of friction, it is $F = f P$. It is distinguishable from the friction of pressure in this; that like impact, it acts only during a very short time. The changes of velocity produced by it are not immeasurably small, for the pressure P, and consequently the part of it $f P$, is generally very great. If we represent the impinging mass by M, and the normal acceleration generated by the pressure P, by p, we shall then have $P = Mp$, and hence $F = f Mp$, as well as the retardation or the negative acceleration due to friction during the impact $\frac{F}{M} = f p$; *i. e.* f times as great as the normal pressure. But the two pressures have equal durations; hence, therefore, *the change of velocity effected by friction is f times as great as the change of the normal velocity effected by impact.*

Fig. 326.



In the case where a body impinges upon an immovable mass BB at the angle of incidence α, Fig. 326, the change in the normal velocity from the former paragraph is $w = c \cos. \alpha\, (1 + \sqrt{\mu})$; hence, the change in the tangential velocity effected by friction

$$= f w = f c\, (1 + \sqrt{\mu}) \cos. \alpha.$$

The lateral velocity, therefore, after impact $c \sin. a$, passes into

$$c \sin. \alpha - fc\, (1 + \sqrt{\mu}) \cos. \alpha = [\sin. \alpha - f \cos. \alpha\, (1 + \sqrt{\mu})\,]\, c,$$

and in the case of perfectly elastic bodies $= (\sin. \alpha - 2 f \cos. \alpha)\, c$, and in that of inelastic $= (\sin. \alpha - f \cos. \alpha)\, c$.

Bodies very often have a rotation about their centre of gravity from the effect of the friction of impact, or the motion of rotation, if present before impact, becomes consequently changed. If the moment of inertia of the round body A about its centre of gravity $S = My^2$, and the radius of gyration $SC = a$, we shall have the mass of the body reduced to the point of contact $C = \frac{My^2}{a^2}$; hence the acceleration of rotation generated by the friction F is:

$$p_1 = \frac{F}{My^2 \div a^2} = \frac{f M p}{My^2 \div a^2} = f p \cdot \frac{a^2}{y^2},$$

and the correspondent change of velocity:

$$w_1 = f \frac{a^2}{y^2} \cdot w = f \frac{a^2}{y^2} (1 + \sqrt{\mu})\, c \cos. \alpha.$$

In a cylinder $\frac{a_2}{y^2} = 2$, and in a sphere $= \frac{5}{2}$, hence the change in the

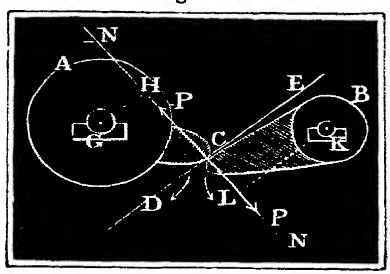
velocity of rotation generated by the impact against the plane is:

$$w_1 = 2f(1+\sqrt{\mu})\,c\,cos.\,\alpha \text{ and } = \frac{5}{2}f(1+\sqrt{\mu})\,c\,cos.\,\alpha.$$

Example. If a billiard-ball, with a 15 feet velocity and at an angle of incidence $\alpha = 45^\circ$, strike against the cushion, what motion will it take after impact? If we put for the $\sqrt{\mu}$ the mean value 0,55, we shall have the lateral normal velocity after impact $= -\sqrt{\mu}\,c\,cos.\,\alpha = -0{,}55\,.\,15\,.\,cos.\,45^\circ = -8{,}25\,.\sqrt{\tfrac{1}{2}} = -5{,}833$ ft. and if with Coriolis we take $f = 0{,}20$, we shall then obtain the lateral velocity parallel to the cushion $= c\,sin.\,\alpha - f(1+\sqrt{\mu})\,c\,cos.\,\alpha = (1-0{,}20\,.\,1{,}55)\,10{,}607 = 0{,}69\,.\,10{,}607 = 7{,}319$ feet, and for the angle of reflexion ϕ:
$tang.\,\phi = \frac{7{,}319}{5{,}833} = 1{,}2548$; therefore, $\phi = 51^\circ\,27'$, and the velocity after impact will remain $= \frac{5{,}833}{cos.\,51^\circ\,27'} = 9{,}360$ feet. Besides, the ball will have further a velocity of rotation $\frac{5}{2}f\,.\,1{,}55\,.\,10{,}607 = 8{,}220$ feet about its vertical line of gravity. Since the ball moves with a rolling and not a sliding motion, we must assume that besides its progressive velocity $c = 15$ feet, it possesses an equal amount of velocity of rotation, and this may likewise be resolved into the components $c\,cos.\,\alpha = 10{,}607$ and $c\,sin.\,\alpha = 10{,}606$. The first component answers to a rotation about an axis parallel to the cushion, and passes into $c\,cos.\,\alpha - \frac{5}{2}f(1+\sqrt{\mu})\,c\,cos.\,\alpha = 10{,}607 - 8{,}220 = 2{,}387$ feet, the other component $c\,sin.\,\alpha = 10{,}607$ feet answers to a rotation about an axis normal to the cushion, and remains uniform.

§ 267. *Rotary Bodies.*—If two bodies, A and B, capable of rotating about two fixed axes G and K, Fig. 327, strike against each other, changes of velocity ensue, which may be determined from the moments of inertia $M_1y_1^2$ and $M_2y_2^2$ of the masses of these bodies about the fixed axes, with the assistance of the formulæ already found. If the perpendiculars GH and KL, let fall from the axis of rotation upon the line of impact, are a_1 and a_2, we then have the inert masses reduced to the points H and L where the perpendiculars meet the line of impact $= \frac{M_1y_1^2}{a_1^2}$ and $\frac{M_2y_2^2}{a_2^2}$, and if these values be substituted for M_1 and M_2 in the formula for free centric impact, we obtain the changes of the velocity of the points H and L (§ 264)

Fig. 327.



$$= (c_1-c_2)\frac{M_2y_2^2 \div a_2^2}{M_1y_1^2 \div a_1^2 + M_2y_2^2 \div a_2^2}(1+\sqrt{\mu})$$

$$= (c_1-c_2)\frac{M_2y_2^2a_1^2}{M_1y_1^2a_2^2 + M_2y_2^2a_1^2}(1+\sqrt{\mu}) \text{ and}$$

$$(c_1-c_2)\frac{M_1y_1^2 \div a_1^2}{M_1y_1^2 \div a_1^2 + M_2y_2^2 \div a_2^2}(1+\sqrt{\mu})$$

$$= (c_1-c_2)\frac{M_1y_1^2a_2^2}{M_1y_1^2a_2^2 + M_2y_2^2a_1^2}(1+\sqrt{\mu}),$$

c_1 and c_2 representing the velocity of these points before impact.

But if we introduce the angular velocities, and represent these be-

fore impact by ε_1 and ε_2, and after impact by ω_1 and ω_2, we shall have to put $c_1 = a_1\, \varepsilon_1$, $c_2 = a_2\, \varepsilon_2$, and shall obtain for the loss in angular velocity of the impinging body:

$$= a_1\,(a_1\, \varepsilon_1 - a_2\, \varepsilon_2)\, \frac{M_2 y_2^{\,2}}{M_1 y_1^{\,2} a_2^{\,2} + M_2 y_2^{\,2} a_1^{\,2}}\, (1 + \sqrt{\mu}),$$

and for the body impinged upon, the gain of the same:

$$= a_2\,(a_1\, \varepsilon_1 - a_2\, \varepsilon_2)\, \frac{M_1 y_1^{\,2}}{M_1 y_1^{\,2} a_2^{\,2} + M_2 y_2^{\,2} a_1^{\,2}}\, (1 + \sqrt{\mu}),$$

consequently the angular velocities themselves, after impact, will be:

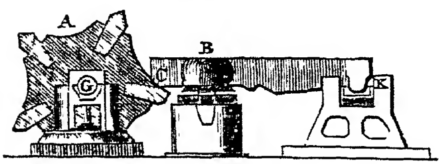
$$\omega_1 = \varepsilon_1 - a_1\,(a_1\, \varepsilon_1 - a_2\, \varepsilon_2)\,(1 + \sqrt{\mu})\, \frac{M_2 y_2^{\,2}}{M_1 y_1^{\,2} a_2^{\,2} + M_2 y_2^{\,2} a_1^{\,2}} \text{ and}$$

$$\omega_2 = \varepsilon_2 + a_2\,(a_1\, \varepsilon_1 - a_2\, \varepsilon_2)\,(1 + \sqrt{\mu})\, \frac{M_1 y_1^{\,2}}{M_1 y_1^{\,2} a_2^{\,2} + M_2 y_2^{\,2} a_1^{\,2}}.$$

If both bodies are perfectly elastic, we shall have $\mu = 1$, therefore $1 + \sqrt{\mu} = 2$, and if inelastic, $\mu = 0$, therefore $1 + \sqrt{\mu} = 1$. In the latter case, the loss of *vis viva* produced by impact

$$= (a_1\, \varepsilon_1 - a_2\, \varepsilon_2)^2 \cdot \frac{M_1 y_1^{\,2} \cdot M_2 y_2^{\,2}}{M_1 y_1^{\,2} a_2^{\,2} + M_2 y_2^{\,2} a_1^{\,2}}.$$

Fig. 328.



Example. The armed axle AG, Fig. 328, has the moment of inertia about its axis of rotation G, $= M_1\, y_1^{\,2} = 40000 \div g$, and the tilt hammer BK one about its axis K, $= 150000 \div g$; the arm GC of the axle is 2 feet, and the arm KC of the hammer 6 feet, and the angular velocity of the axle at the moment of impact on the hammer $= 1,05$ feet; what is this velocity after impact, and what effect is lost at each blow, if there is an entire absence of elasticity? The angular velocity of the axle sought is:

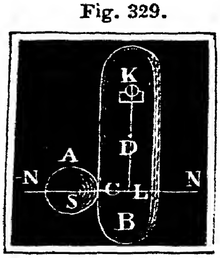
$$\omega_1 = 1,05 - \frac{4 \,.\, 1,05 \,.\, 150000}{40000 \,.\, 36 + 150000 \,.\, 4} = 1,105 \left(1 - \frac{60}{204}\right) = 1,05 \,.\, 0,706$$

$= 0,741$ feet, and that of the hammer $= \frac{2 \,.\, 6 \,.\, 1,05 \,.\, 4}{204} = 0,247$ feet, *i. e.*, only one-third that of the axle. The loss of mechanical effect by each blow is:

$$L = \frac{(2 \,.\, 1,05)^2}{2g} \cdot \frac{4 \,.\, 1,05 \,.\, 150000}{40000 \,.\, 36 + 150000 \,.\, 4} = 0,0155 \,.\, (2,1)^2 \,.\, \frac{600000}{144 + 60}$$

$$= 0,0155 \,.\, 4,41 \,.\, \frac{150000}{51} = 201 \text{ ft. lbs.}$$

Fig. 329.



§ 268. A body A, in a state of free and progressive motion, Fig. 329, impinges against a body BCK, capable of rotating about a fixed axis K, the velocities after impact may be found, if, in place of $a_1\, \varepsilon_1$ and $a_1\, \omega_1$ in the formula of the preceding paragraph, we put the progressive velocities c_1 and v_1, and instead of $\frac{M_1 y_1^{\,2}}{a_1^{\,2}}$ the inert mass M_1 of the first body, the other denominations remaining the same. Hence, the velocity of the first mass after impact is:

$$v_1 = c_1 - (c_1 - a_2\,\varepsilon_2)\,(1 + \sqrt{\mu}) \cdot \frac{M_2 y_2^{\,2}}{M_1 a_2^{\,2} + M_2 y_2^{\,2}},$$

and the angular velocity of the second:

$$\omega_2 = \varepsilon_2 + a_2\,(c_1 - a_2\,\varepsilon_2)\,(1 + \sqrt{\mu}) \cdot \frac{M_1}{M_1 a_2^{\,2} + M_2 y_2^{\,2}}.$$

If the mass M_2 be at rest, therefore $\varepsilon_2 = 0$, we have:

$$v_1 = c_1 - c_1\,(1 + \sqrt{\mu}) \cdot \frac{M_2 y_2^{\,2}}{M_1 a_2^{\,2} + M_2 y_2^{\,2}} \text{ and}$$

$$\omega_2 = a_2\, c_1\,(1 + \sqrt{\mu}) \cdot \frac{M_1}{M_1 a_2^{\,2} + M_2 y_2^{\,2}}.$$

If, on the other hand, M_1 is at rest, that is, the oscillating body the impinging one, we shall have $c_1 = 0$, and hence

$$v_1 = a_2\,\varepsilon_2\,(1 + \sqrt{\mu}) . \frac{M_2 y_2^{\,2}}{M_1 a_2^{\,2} + M_2 y_2^{\,2}} \text{ and}$$

$$\omega_2 = \varepsilon_2 \left(1 - (1 + \sqrt{\mu})\,\frac{M_1 a_2^{\,2}}{M_1 a_2^{\,2} + M_2 y_2^{\,2}}\right).$$

The velocity communicated by impact to another at rest, depends not only on the velocity of impact and of the masses of the bodies, but also on the distance $KL = a_2$ at which the direction of impact NN is distant from the axis K of the rotary body. If the free mass be the impinging one, the rotary mass will assume the angular velocity

$$\omega_2 = c_1\,(1 + \sqrt{\mu})\,\frac{M_1 a_2}{M_1 a_2^{\,2} + M_2 y_2^{\,2}},$$

and if the oscillating mass strike against the free, this will acquire the velocity

$$v_1 = \varepsilon_2\,(1 + \sqrt{\mu})\,\frac{M_2 y_2^{\,2} \cdot a_2}{M_1 a_2^{\,2} + M_2 y_2^{\,2}},$$

but both velocities will be so much the greater, the greater

$\dfrac{a_2}{M_1 a_2^{\,2} + M_2 y_2^{\,2}}$ or $\dfrac{1}{M_1 a_2 + \dfrac{M_2 y_2^{\,2}}{a_2}}$ is, and therefore the less

$M_1 a_2 + M_2 \dfrac{y_2^{\,2}}{a_2}$ is.

If for a_2 we put $a \pm x$, when x is very small, we shall obtain the value of the last impression:

$$M_1\,(a \pm x) + \frac{M_2 y_2^{\,2}}{a \pm x} = M_1 a \pm M_1 x + \frac{M_2 y_2^{\,2}}{a}\left(1 \mp \frac{x}{a} + \frac{x^2}{a^2} \pm \ldots\right)$$

or in consequence of the smallness of the powers of x,

$$= M_1 a + \frac{M_2 y_2^{\,2}}{a} \pm \left(M_1 - \frac{M_2 y_2^{\,2}}{a^2}\right) x + \ldots$$

If now a correspond to the least of all the values of $M_1\,a_2 + \dfrac{M_2 y_2^{\,2}}{a_2^{\,2}}$, the member $\pm \left(M_1 - \dfrac{M_2 y_2^{\,2}}{a^2}\right) x$ will disappear, because the addition of the quantity (x) will give to it a different sign to that of a diminutive $(-\,x.)$ Therefore:

$\left(M_1 - \frac{M_2 y_2^2}{a^2}\right) x$ must be $= 0$, *i. e.* $\frac{M_2 y_2^2}{a^2} = M_1$, consequently:

$$a = \sqrt{\frac{M_2 y_2^2}{M_1}} = y_2 \sqrt{\frac{M^2}{M_1}}.$$

If at this distance one body impinge against the other, then will th latter take the greatest velocity, and be in fact

$$\omega = c_1 (1 + \sqrt{\mu}) \frac{1}{2\, y_2} \sqrt{\frac{M_1}{M_2}},$$

in the case where the rotary body is impinged upon; and

$$v = \tfrac{1}{2}\, y_2 \varepsilon_2 (1 + \sqrt{\mu}) \sqrt{\frac{M_2}{M_1}},$$

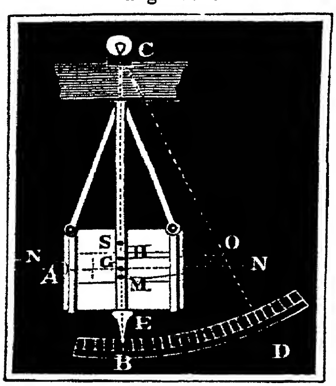
when the free body receives the blow.

The point D in the line of impact of the distance corresponding t the greatest velocity, or of the arm a, is sometimes improperly calle the centre of impact, but more properly the *point of impact.*

Example. What position has the point of impact if the free mass consist of an iro sphere of 16 lbs. weight, and the rotary mass have a moment of inertia of 100 $\div\, g$? The distance of this point from the fixed axis of the last body: $a = \sqrt{\frac{1000}{16}} = \sqrt{62{,}5} = 7{,}906$ feet. If the impact be inelastic, and the block strike agains the sphere with the velocity $\varepsilon = 3$ feet, the latter will receive the velocity $v = \frac{3}{2} \cdot 7{,}906 = 11{,}86$ feet.

§ 269. *Ballistic Pendulum.*—An application of the laws laid down is found in the theory of the *ballistic pendulum*, or the *pendulum of Robins.* I consists of a mass MH, turning about a horizontal axis C, Fig. 330, which is se into oscillating motion by a ball projectec against it, which serves for the measurement of its velocity. That as inelastic a blow as possible may ensue, there is ar opening made on the further side, which from time to time is filled by fresh wood or clay, &c. The ball remains after each projection sticking in this mass, and oscillating in common with the whole body. For the measurement of the velocity of the ball, it is requisite to know the angle of elongation of this pendulum, on which account there is further a graduated arc BD applied, and an index E fixed to the centre of gravity of the pendulum, which slides along with the former.

Fig. 330.



From the foregoing paragraph, the angular velocity of the ballistic pendulum after the impact of the ball is: $\omega = \frac{M_1 a_2 c_1}{M_1 a_2^2 + M_2 y_2^2}$, if M_1 is the mass of the ball, $M_2 y_2^2$ the moment of inertia of the pendulum,

c_1 the velocity of the ball, and a_2 the arm CG of the impact, or the distance of the line of impact NN from the axis of revolution of the pendulum. If the distance CM of the centre of oscillation M of the entire mass, together with the ball from the centre of suspension C, *i. e.* the length of the simple pendulum, which oscillates in equal times with the ballistic, $= l$, and the angle of elongation $BCD = \alpha$, we have the height of the isochronously oscillating simple pendulum:

$$h = CM - CH = l - l\,cos.\,\alpha = l(1 - cos.\,\alpha) = 2l = \left(sin.\,\frac{\alpha}{2}\right)^2;$$

and hence the velocity at the lowest point of its path:

$$v = \sqrt{2gh} = 2\sqrt{gl}\,sin.\,\frac{\alpha}{2}.$$

or the corresponding angular velocity:

$$\omega = \frac{v}{l} = 2\sqrt{\frac{g}{l}}\,.\,sin.\,\frac{\alpha}{2}\,.$$

By equating these two values it will follow that the angular velocity:

$$c_1 = \frac{M_1 a_2^2 + M_2 y_2^2}{M_1 a_2}\,.\,2\sqrt{\frac{g}{l}}\,.\,sin.\,\frac{\alpha}{2}.$$

But now, according to the theory of the simple pendulum,

$$l = \frac{\text{moment of inertia}}{\text{statical moment}} = \frac{M_1 a_2^2 + M_2 y_2^2}{(M_1 + M_2)\,s},$$

s being the distance of the centre of gravity S from the axis of revolution, hence

$$M_1 a_2^2 + M_2 y_2^2 = (M_1 + M_2)\,sl, \text{ and}$$

$$c_1 = 2\left(\frac{M_1 + M_2}{M_1}\,.\right)\frac{s}{a_2}\sqrt{gl}\,.\,sin.\,\frac{\alpha}{2}.$$

If the pendulum makes n oscillations per minute, the time of oscillation

$$\pi\sqrt{\frac{l}{g}} = \frac{60''}{n}, \text{ hence } \sqrt{gl} = \frac{60''\,.\,g}{n\pi},$$

and the required velocity of the ball

$$c_1 = \frac{M_1 + M_2}{M_1}\,.\,\frac{120\,gs}{n\,\pi\,a_2}\,.\,sin.\,\frac{\alpha}{2}.$$

Example. If a ballistic pendulum, of 3000 lbs. weight, whose angle of elongation amounts to 15°, is set into oscillation by the projection of a 6 lbs. ball, if, further, the distance s of the centre of gravity from the axis $= 5$ feet, and that of the line of projection from this axis $= 5\frac{1}{2}$ feet, and lastly, the number of oscillations per minute $n = 40$, from the above formula the velocity of the ball at the moment of the impact will be:

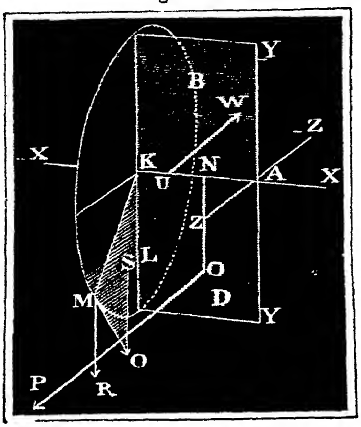
$$c = \frac{3006}{6}\,.\,\frac{120\,.\,31{,}25\,.\,5}{40\,.\,3{,}1416\,.\,5{,}5}\,sin.\,7\tfrac{1}{2}^\circ = \frac{501\,.\,3750\,.\,sin.\,7^\circ\,30'}{44\,.\,3{,}1416} = 1774 \text{ feet.}$$

§ 270. *Centre of Percussion.*—If a body turning about a fixed axis C is impinged upon by another, a reaction from the blow will generally take place upon the axis of the body, which is dependent principally upon the distance between the direction of the impact and that of the axis. Let us determine this reaction or this axial pressure in the simple case, when the direction of the blow is perpendicular to the

plane passing through the axis of revolution and the centre of gravity of the body.

Let BD be a plane of gravity passing through the axis of revolution $X\overline{X}$ of the body in Fig. 331, $Y\overline{Y}$ a second perpendicular axis in this plane, and $Z\overline{Z}$ a third axis, at right angles to this plane of gravity. An element M_1 of the body is determined by the co-ordinates $AK = x_1$, $KL = y_1$, and $LM = z_1$, in this system of axes intersecting in the point A, and another element by the co-ordinates x_2, y_2, z_2, &c. If $\varkappa$ be the angular acceleration, we shall have the force of inertia of the element $M_1 : Q_1 = M_1 . \varkappa . \overline{KM}$, and if these be resolved into the component forces R parallel, and S at right angles to the imaginary plane of gravity, the similarity of the triangles KML and QMR or MQS, gives $R = M_1 . \varkappa z_1 = \varkappa M_1 z_1$, and $S = \varkappa M_1 y_1$. From this the aggregate of the component forces parallel to the plane $= \varkappa (M_1 z_1 + M_2 z_2 + ..)$ and that of the component forces at right angles to this plane:

Fig. 331.



$$\varkappa (M_1 y_1 + M_2 y_2 + ..).$$

Since the plane BD passes through the centre of gravity, the sum of the moments $M_1 z_1 + M_2 z_2 + .. = 0$, hence there remains only the sum of the forces $\varkappa (M_1 y_1 + M_2 y_2 + ..)$. If, now, P be a blow or impact, and W the resistance, or the reaction of the axis, we shall, in the first place, have to put:

$$P = W + \varkappa (M_1 y_1 + M_2 y_2 ...).$$

The statical moment of the force:

$$Q_1 = M_1 \varkappa . \overline{KM} = M_1 \varkappa . \overline{KM} . \overline{KM} = \varkappa . M_1 \overline{KM}^2,$$

or the distance KM represented by $r_1 = \varkappa M_1 r_1^2$, the moment of the force of another particle of the mass $= \varkappa M_2 r_2^2$, &c., hence the statical moment of the entire inertia $= \varkappa (M_1 r_1^2 + M_2 r_2^2 + ...)$. If now we put the distance NO of the direction of the impact from the direction of the axis $= b$, we shall have the moment of percussion P about $X\overline{X} = Pb$, whilst that of $W = 0$; we may hence also put:

$$Pb = \varkappa (M_1 r_1^2 + M_2 r_2^2 + ..),$$

and obtain by elimination of $\varkappa$ from both equations:

$$P = W + \frac{Pb (M_1 y_1 + M_2 y_2 + ..)}{M_1 r_1^2 + M_2 r_2^2 + ..}, \text{ i. e. the reaction sought:}$$

$$W = P \left(1 - \frac{b (M_1 y_1 + M_2 y_2 + ..)}{M_1 r_1^2 + M_2 r_2^2 + ..}\right).$$

If, lastly, we represent the distance AN of the direction of the impact from the axis $Y\overline{Y}$ by a, and the distance AU of the point of application U of the reaction W from the initial point by u, we shall have further the moment $Pa =$ mom. $Wu +$ mom. $\varkappa (M_1 x_1 y_1 + M_2 x_2 y_2 + ..)$,

and the distance of the point of application sought will be:

$$u = \frac{Pa - \varkappa\,(M_1x_1y_1 + M_2x_2y_2 + \,.\,.)}{W},\ i.\,e.$$

$$u = \frac{a\,(M_1r_1^{\,2} + M_2r_2^{\,2} + \,.\,.) - b\,(M_1x_1y_1 + M_2x_2y_2 + \,.\,.)}{M_1r_1^{\,2} + M_2r_2^{\,2} + \,.\,. - b\,(M_1y_1 + M_2y_2 + \,.\,.)}$$

The reaction $W = 0$, if $b\,(M_1y_1 + M_2y_2 + \,.\,.) = M_1r_1^{\,2} + M_2r_2^{\,2} + \,.\,.$,

$$i.\,e.\ 1.\ b = \frac{M_1r_1^{\,2} + M_2r_2^{\,2} + \,.\,.}{M_1y_1 + M_2y_2 + \,.\,.} = \frac{\text{moment of inertia}}{\text{statical moment}},$$

and also its moment $= 0$, if

$$Pa = \varkappa\,(M_1x_1y_1 + M_2x_2y_2 + \,.\,.),\ i.\,e.$$

$$2.\ a = \frac{M_1x_1y_1 + M_2x_2y_2 + \,.\,.}{M_1y_1 + M_2y_2 + \,.\,.}.$$

The point O determined by these co-ordinates a and b, in the plane of gravity containing the fixed axis, is called the *centre of percussion.* Every blow passing through this point, and at right angles to the plane of gravity, is completely taken up by the mass, without leaving any residuary effect upon the axis, or producing any pressure. The formula (1) shows that the centre of percussion is at the same distance from the axis of revolution (compare § 251) as the centre of suspension.

That a hammer may not jar by its blow the hand which holds it, or react upon the wrist about which it turns, it is requisite that the blow pass through the centre of percussion.

Examples.—1. In a prismatic bar CA, Fig. 332, which turns about one of its extreme points, the centre of percussion lies about $CO = b = \frac{\frac{1}{3}l^2}{\frac{1}{2}l} = \frac{2}{3}\,l = \frac{2}{3}\,CA$ from the axis. If, therefore, the bar be fixed at one extremity, and be struck at a point O at the distance $CO = \frac{2}{3}\,CA$, then no jar will be felt.—2. In a parallelopiped BDE, Fig. 333, which

Fig. 332.

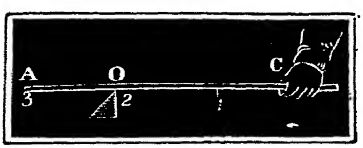
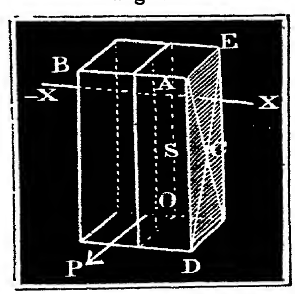


Fig. 333.

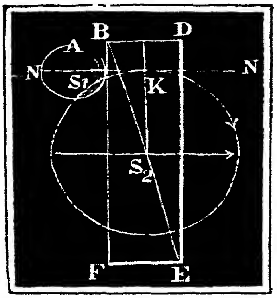


turns about an axis $\overline{XX}$ running parallel to its four sides, and distant about $SA = s$ from the centre of gravity, the distance AO of the centre of percussion O from the axis $b = \frac{s^2 + \frac{1}{3}d^2}{s}$, where d is the semi-diagonal of the lateral surfaces through which the axis $\overline{XX}$ passes (§ 220). If the force of the blow P were to pass through the centre of gravity, the reaction would be:

$$W = P\left(1 - s\cdot\frac{s}{s^2 + \frac{1}{3}d^2}\right) = P\left(1 - \frac{s^2}{s^2 + \frac{1}{3}d^2}\right) = \tfrac{1}{3}\cdot\frac{Pd^2}{s^2 + \frac{1}{3}d^2} = \frac{Pd^2}{3s^2 + d^2}.$$

§ 271. *Excentric Impact.*—Lastly, let us further investigate a simple case of excentric impact, when both masses are perfectly free. When two bodies A and BE, Fig. 334, impinge upon each other so that the direction of impact NN passes through the centre of gravity S_1 of the one body, and beyond the centre of gravity S_2 of the other body, the impact with respect to the first body is centric, and with respect to the other, excentric. The effects of this excentric impact may be found from the proposition, § 214, if we assume first, that the second body is free, and that the direction of impact passes through the centre of gravity S_2, and secondly, that this body is fixed at its centre of gravity, and that the percutient force acts as a force of revolution. If now c_1 be the initial velocity of A, c_2 that of the centre of gravity of BE, and if the two velocities pass into v_1 and v_2 from the effect of the blow, there will remain as in § 256,

Fig. 334.



$$M_1 v_1 + M_2 v_2 = M_1 c_1 + M_2 c_2.$$

If further ε be the initial angular velocity of the body BE, and in its revolution about the axis passing through the centre of gravity, and perpendicular to the plane NNS_2, and if this velocity pass from the effect of the blow into ω, and $M_2 y_2^2$ represent the moment of inertia of this body about S_2, or s the distance S_2K of the centre of gravity S_2 from the direction of impact, we shall then also have

$$M_1 v_1 + \frac{M_2 y_2^2}{s^2} \cdot s\omega = M_1 c_1 + \frac{M_2 y_2^2}{s^2} s\varepsilon.$$

If both bodies are inelastic, then the points of contact of the two will have at the end equal velocities, therefore, further v_1 will $= v_2 + s\omega$. If we determine from the above equations v_2 and ω by v_1, and put the values obtained into the last equation, we shall then obtain:

$$v_1 = \frac{M_1(c_1 - v_1)}{M_2} + c_2 + \frac{M_1 s^2 (c_1 - v_1)}{M_2 y_2^2} + s\varepsilon,$$

and from this may be determined the loss in velocity of the first body:

$$c_1 - v_1 = \frac{M_2 y_2^2 (c_1 - c_2 - s\varepsilon)}{(M_1 + M_2) y_2^2 + M_1 s^2},$$

the gain in progressive velocity of the second:

$$v_2 - c_2 = \frac{M_1 y^2 (c_1 - c_2 - s\varepsilon)}{(M_1 + M_2) y_2^2 + M_1 s^2},$$

and the gain in angular velocity:

$$\omega - \varepsilon = \frac{M_1 s (c_1 - c_2 - s\varepsilon)}{(M_1 + M_2) y_2^2 + M_1 s^2}.$$

For the case of perfectly elastic impact, these values are double; and for that of imperfectly elastic impact, they are $(1 + \sqrt{\mu})$ times as great.

Example. An iron ball A, of 65 lbs. weight, strikes a parallelopiped BE of fir, originally at rest, with a 36 feet velocity; the length of this body is 5 feet, its breadth 3 feet, and thickness 2 feet, and the direction of the impact $\overline{NN}$ deviates by $S_2 K = s = 1\frac{3}{4}$ feet from the centre of gravity S_2, then the following velocities after impact are given. The specific gravity of fir may be taken $= 0{,}45$, the weight of the parallelopiped is therefore $= 5 \times 3 \times 2 \times 62{,}5 \times 0{,}45 = 843{,}75$ lbs. The square of the semi-diagonal of the lateral surface parallel to the direction of impact is:

$$y^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{2}{2}\right)^2 = 7{,}25,$$

hence the velocity of the ball after impact is:

$$v_1 = c_1 - \frac{M_2 y^2 c_1}{(M_1 + M_2)\, y^2 + M_1 s^2} = 36\left(1 - \frac{843{,}75\,.\,7{,}25}{956\,.\,7{,}25 + 65\,.\,1{,}75^2}\right)$$

$$= 36\left(1 - \frac{843{,}75\,.\,7{,}25}{7130{,}06}\right) = 36\,(1 - 0{,}958) = 1{,}512 \text{ feet}:$$

further, the velocity of the centre of gravity of the parallelopiped:

$$v_2 = \frac{M_1 y^2 c_1}{(M_1 + M_2)\, y^2 + M_1 s^2} = \frac{65\,.\,7{,}25\,.\,36}{7130{,}06} = 2{,}379 \text{ feet}:$$

lastly, the angular velocity of this body is:

$$w = \frac{M_1 s c}{(M_1 + M_2)\, y^2 + M_1 s^2} = \frac{65\,.\,1{,}75\,.\,36}{7130{,}06} = 0{,}574 \text{ feet.}$$

SECTION V.

STATICS OF FLUID BODIES.

CHAPTER I.

ON THE EQUILIBRIUM AND PRESSURE OF WATER IN VESSELS.

§ 272. *Fluidity.*—We regard *fluid bodies* as systems of material points, whose cohesion is so feeble, that the smallest forces are sufficient to effect a separation, and to move them amongst each other (§ 59). Many bodies in nature, such as air, water, &c., possess this property of fluidity in a high degree; other bodies, on the contrary, such as oil, fat, soft earth, &c., are fluid in a low degree. The former are called *perfectly fluid*, the latter *imperfectly fluid bodies*. Certain bodies, as, for instance, paste, are intermediate between solid and fluid bodies.

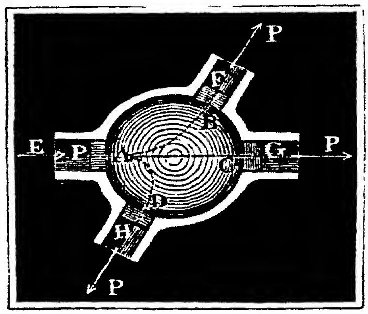
Perfectly fluid bodies, of which only we shall subsequently speak, are at the same time perfectly elastic, *i. e.*, they may be compressed by external forces, and will perfectly resume their former volume after the withdrawal of these forces. The amount of the change of volume corresponding to a certain pressure, is different for different fluids; in *liquid bodies* this is scarcely perceptible, while in *aëriform bodies*, which, on this account, are also called elastic fluids, it is very great. This slight degree of compressibility of liquid bodies is the reason why in most investigations in hydrostatics (§ 63) they are considered and treated as incompressible or inelastic. As water, of all liquids, is the one most generally diffused, and the most useful for the purposes of life, it is taken as the representant of all these fluids, and in the investigations of the mechanics of fluids, water only is spoken of, whilst it is tacitly understood that the mechanical properties of other liquids are the same as those of water.

From a similar reason in the mechanics of the elastic fluid bodies ordinary atmospheric air is only spoken of.

Remark.—A column of water of one square inch transverse section is compressed by a weight of 15 lbs, which corresponds to the atmospheric pressure, by about 0,00005 or 50 millionths of its volume, while the same column of air under this pressure would be compressed to one half of its original volume.

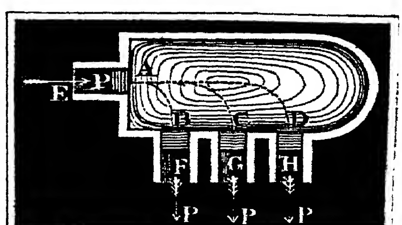
§ 273. *Principle of Equality of Pressures.*—The characteristic property of fluids, which essentially distinguishes them from solid bodies, and which serves as a basis of the laws of the equilibrium of fluid bodies, *is the capability of transmitting the pressure which is exerted upon a part of the surface of the fluid in all directions unchanged.* The pressure on solids is transmitted only in its proper direction (§ 83); while, on the other hand, when water is pressed on one side, a tension takes place in the entire mass, which exerts itself on all sides, and may be observed at all parts of the surface. To satisfy ourselves of the correctness of this law, we may make use of an apparatus filled with water, as is shown in the horizontal section in Fig. 335. The tubes AE and BF, &c., equally distant, and at an equal height above the horizontal base, are closed by perfectly movable and accurately fitting pistons; the water presses, therefore, by its weight, as strongly against the one piston as against the other. Let us do away with this pressure, and regard the water as devoid of weight. Let us press the one piston with a certain pressure P against the water, this pressure will then be transmitted by the water to the other pistons B, C, D, and for the restoration of equilibrium, or to prevent the pushing back of these pistons, it is requisite that an equal and opposite pressure P act against each of these pistons. We are, therefore, justified in assuming, that the pressure P, acting upon a point A of the surface of the mass of water, produces in it a tension, and not only transmits this in the straight line AC, but also in every other direction BF, DH, &c., to every equal area of the surface C, B, D.

Fig. 335.



If the axes of the tubes BF, CG, &c., Fig. 336, are parallel to each other, the pressures which act upon their pistons may be united by addition into a single pressure; if n be the number of the pistons, then the aggregate pressure upon these amounts to $P_1 = nP$, and in the case represented in the figure $P_1 = 3P$. But now the areas F_1 of the pressed surfaces B, C, D, are equal to n times the pressed surface F of the one piston, hence n may not only be put $= \frac{P_1}{P}$, but also $= \frac{F_1}{F}$, therefore $\frac{P_1}{P} = \frac{F_1}{F}$.

Fig. 336.



If the tubes B, C, D, form a single one, as in Fig. 337, and if we close it by a single piston, F_1 then becomes a single surface, and P_1

is the pressure acting upon it, hence there follows this general law, *the pressure which a fluid body exerts upon different parts of the sides of a vessel, is proportional to the area of these parts.*

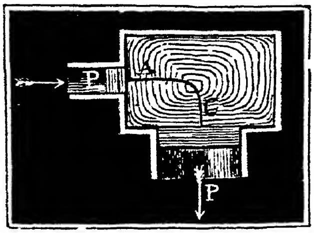


Fig. 337.

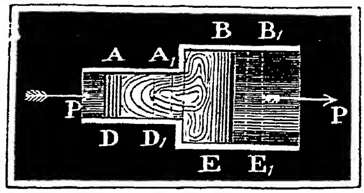


Fig. 338.

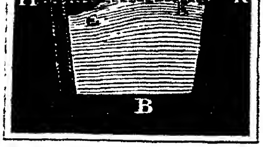
This law corresponds also to the principle of virtual velocities. If the piston $AD = F$, Fig. 338, moves inwards through a space $AA_1 = s$, it then presses the column of water Fs from its tube, and if the piston $BE = F_1$, it passes outwards through the space $BB_1 = s_1$, it then leaves a space $F_1 s_1$ behind. But since we have supposed that mass of water neither allows of expansion nor compression, its volume then by this motion of the piston must remain unaltered, that is, the increase Fs must be equal to the decrease $F_1 s_1$. But the equation $F_1 s_1 = Fs$ gives $\frac{F_1}{F} = \frac{s}{s_1}$, and by combining this proportion with the proportion $\frac{P_1}{P} = \frac{F_1}{F}$, it follows that $\frac{P_1}{P} = \frac{s}{s_1}$, hence, therefore, the mechanical effect $P_1 s_1 =$ mechanical effect Ps (§ 80).

Example. If the piston AD has a diameter of $1\frac{1}{2}$ inches, and the piston BE one of 10 inches, and each is pressed by a force P of 36 lbs. upon the water, this piston exerts a pressure $P_1 = \frac{F}{F_1} P = \frac{10^2}{1,5^2} \cdot 36 = 1600$ lbs. If the first piston is pushed forwards 6 inches, the second will only go back by $s_1 = \frac{F}{F_1} s = \frac{9 \cdot 6}{400} = \frac{27}{200} = 0,135$ in.

Remark. Numerous applications of this law will come before us in the hydraulic press, or water column machines, in pumps, &c.

§ 274. *The Fluid Surface.*—The gravity inherent in water causes all its particles to tend downwards, and they would actually so move unless this motion were prevented. In order to obtain a coherent mass of water, it is necessary to enclose it in vessels. The water in the vessel ABC, Fig. 339, is then only in equilibrium if its free surface HR is perpendicular to the direction of gravity, and therefore horizontal, for so long as this surface is curved or inclined to the horizon: then there are elementary portions E, F, &c., lying higher, which, from their extreme mobility in virtue of their gravity, slide down on those below them, as if it were on an inclined plane GK.

Fig. 339.

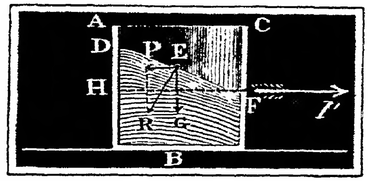


Since the directions of gravity for great distances can no longer be regarded as parallel, we must, therefore, consider the free surface, or the level of water in a large vessel, as for example, in a great lake, no longer as a plane, but as part of a spherical surface.

If any other force than that of gravity act upon the particles of water, the fluid surface in the state of equilibrium, will be perpendicular to the direction of the resultant arising from gravity and the concurrent force.

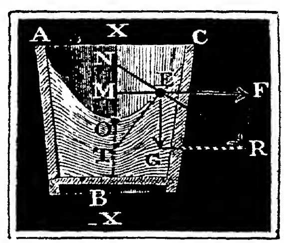
If a vessel ABC, Fig. 340, is moved forward horizontally by a uniformly accelerating force p, the free surface of the water in it will form an inclined plane DF, for in this case every element E of this surface will be impelled downwards by its weight G, and horizontally by its inertia $P = \frac{p}{g} G$, there will then be a resultant R, which will make with the direction of gravity a uniform angle $REG = \alpha$. This angle is at the same time the angle DFH which the surface of the water makes with the horizon. It is determined by $tang. \alpha = \frac{P}{G} = \frac{p}{g}$.

Fig. 340.



If, on the other hand, a vessel ABC, Fig. 341, rotates uniformly about its vertical axis $\overline{XX}$, the surface of the water then forms a hollow surface AOC, whose sections through the axis are parabolic. If ω be the angular velocity of the vessel and the water in it, G the weight of an element of water E, and y its distance ME from the vertical axis, we shall then have for the centrifugal force of this element $F = \omega^2 \frac{Gy}{g}$ (§ 231), and hence for the angle $REG = TEM = \phi$, which the resultant R makes with the vertical or the tangent to the water profile with the horizon:

Fig. 341.

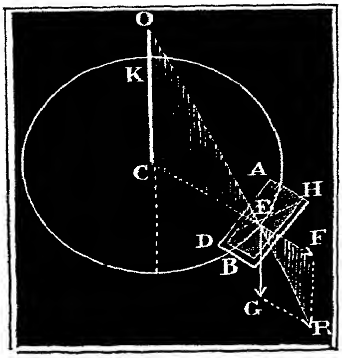


$$tang. \phi = \frac{F}{G} = \frac{\omega^2 y}{g}.$$

From this, therefore, the tangent of the angle which the line of contact makes with this ordinate, is proportional to the ordinate. As this property belongs to the common parabola (§ 144), the vertical section AOC of the surface of water is also a parabola whose axis coincides with the axis of revolution $\overline{XX}$.

If a vessel ABH be moved in a vertical circle, Fig. 342, uniformly about a horizontal parallel axis C, the surface of the water will form in it a cylindrical surface with circular sections DEH. If we prolong the direction of the resultant R of the gravity G, and the centrifugal force F of an element E to the intersection O with the vertical CK passing through the centre of revolution, we shall then

Fig. 342.



obtain the similar triangles ECO and EFR, for which

$$\frac{OC}{EC} = \frac{FR}{EF} = \frac{G}{F},$$

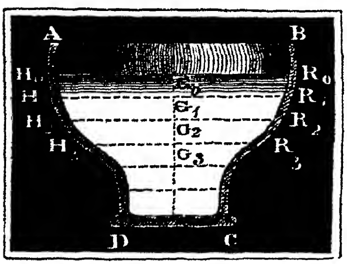
but now, if we put the radius of gyration $EC = y$, and retain the last notation, $F = \frac{\omega^2 Gy}{g}$, it follows that the line

$$CO = \frac{g}{\omega^2} = \frac{32,2}{(3,1416)^2} \left(\frac{30}{u}\right)^2 = \frac{2936}{u^2}$$

if u represents the number of revolutions per minute. As this value of CO is one and the same for all the particles of water, it follows that the components of all the particles forming the section DEH are directed towards O, and hence the section perpendicular to the directions of these forces, is a circle described from O as a centre. According to this, the surfaces of water in the buckets of an overshot wheel form perfect cylindrical surfaces, corresponding to one and the same horizontal axis.

§ 275. *Pressure on the Bottom.*—The pressure of water in a vessel $ABCD$, Fig. 343, immediately under the water level is the least, but becomes greater and greater in proportion to the depth, and is greatest immediately above the bottom. To prove this generally, let us assume that the level of the water H_0R_0, whose area may be F_0, is uniformly pressed by a force P_0, for example, by the superincumbent atmosphere, or by a piston, and let us suppose the whole mass of water divided by horizontal planes, as H_1R_1, H_2R_2, into equally thick strata of water. If now λ be the thickness or the height of such a stratum, and γ the density of water, we shall then have the weight of the first stratum $G_1 = F_0\lambda\gamma$, and hence the entire pressure on the subjacent water: $P_1 = P_0 + F_0\lambda\gamma$. If we divide this pressure by the area F_1 of the following horizontal section H_1R_1, we shall obtain the pressure for each unit of this surface: $p_1 + \frac{P_0}{F_1} = \frac{F_0}{F_1}, \lambda\gamma$, or, since F_1, on account of the infinitely small distance between H_0R_0 and H_1R_1, is infinitely little different from F_0, and may be substituted for this: $p_1 = p_0 + \lambda\gamma$, where p_0 represents the external pressure on the unit of surface. The pressure of the succeeding horizontal section H_2R_2 may be determined as exactly as the pressure of the stratum H_1R_1, if we take into consideration that the initial pressure upon the unit is now $p_1 = p_0 + \lambda\gamma$, whilst it was then only p_0. The pressure in the horizontal stratum H_2R_2 then follows: $p_2 = p_1 + \lambda\gamma = p_0 + \lambda\gamma + \lambda\gamma = p_0 + 2\lambda\gamma$; likewise the pressure in

Fig. 343.



the third stratum $H_3R_3 = p + 3\lambda\gamma$, in the fourth $= p_0 + 4\lambda\gamma$, and in the nth $= p_0 + n\lambda\gamma$. But now $n\lambda$ is the depth $G_0Gn = h$ of the nth stratum below the level of the water, hence the pressure upon each unit of surface in the nth horizontal stratum may be put: $p = p_0 + h\gamma$.

The depth h of an element of surface below the water level, is called the *head of water*, and the pressure of water upon any unit of surface may from this be found, if the externally acting pressure be increased by the weight of a column of water whose base is this unit, and whose height is the head of water.

The head of water h on a horizontal surface, for instance, on the bottom CD, is at all places one and the same; hence the area of this surface $= F$, and the pressure of water against it is: $P = (p_0 + h\gamma) F = Fp_0 + Fh\gamma = P_0 + Fh\gamma$, or if we abstract the outer pressure: $P = Fh\gamma$. *The pressure of water against a horizontal surface is therefore equivalent to the weight of the superincumbent column of water Fh.*

This pressure of water against a horizontal surface—a horizontal bottom, for instance—or against a horizontal part of a lateral wall, is independent of the form of the vessel; whether, therefore, the vessel AC, Fig. 344, be prismatic as a, or wider above than below as b, or wider below than above, as c, or inclined as d, or bulging out as e, &c., the pressure on the bottom will be always equal to the weight of a column of water whose base is the bottom and whose height is the depth of the bottom below the level of the water. As the pressure of water transmits itself on all sides, this law is therefore applicable when the surface, as BC, Fig. 345, is pressed upon from below upwards. Every unit of surface in the stratum lying in BC is pressed by a column of water of the height $HB = RK = h$; consequently, the pressure against $CB = Fh\gamma$, F being the area of the surface.

Fig. 344.

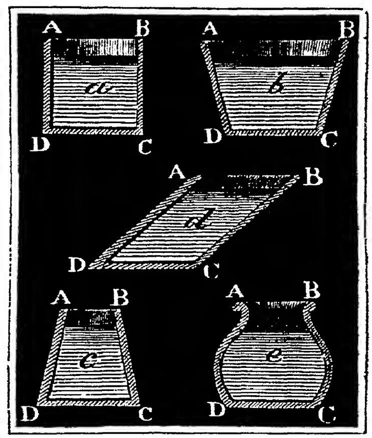


Fig. 345.

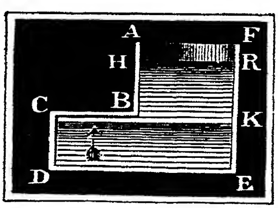
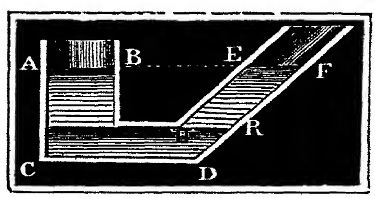


Fig. 346.

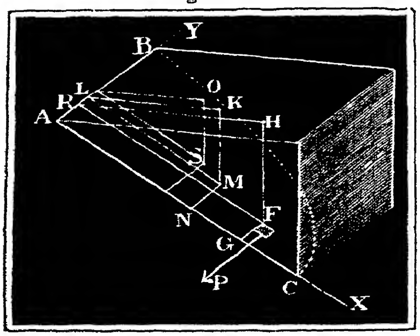


It further follows from this, that the water in tubes communicating with each other ABC and DEF, Fig. 346, when equilibrium subsists, stands equally high, or that the two levels AB and EF are in one and

the same horizontal plane. For the subsistence of equilibrium, it is requisite that the stratum of water HR be as forcibly pressed downwards by the superincumbent column of water ER, as pressed upwards by the mass of water lying below it. But as in both cases the surface pressed is one and the same, so must the head of water in both cases be one and the same, therefore the level AB must stand as high above HR as the level EF.

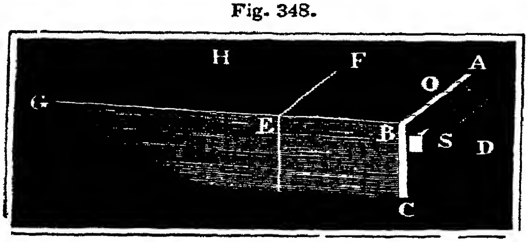
§ 276. *Lateral Pressure.*—The laws found above for the pressure of water against a horizontal surface, are not directly applicable to a plane surface inclined to the horizon; for in this case the heads of water at different places are different. The pressure $p = h\gamma$ on each unit of surface within the horizontal stratum of water, which lies a depth h below the level, acts in all directions (§ 273), and consequently also perpendicular to the fixed lateral walls of the vessel, which (from § 128) perfectly counteract it. If now F_1 be the area of an element of a lateral surface ABC, Fig. 347, and h_1 its head of water FH, we shall then have the normal pressure of the water against it: $P_1 = F_1 . h_1\gamma$; if F_2 be a second element of the surface, and h_2 its head of water, we shall then have the normal pressure on it: $P_2 = F_2h_2\gamma$; and for a third element $P_3 = F_3h_3\gamma$, &c. These normal pressures form a system of parallel forces, whose resultant P is the sum of these pressures; therefore $P = (F_1h_1 + F_2h_2 + \ldots)\gamma$. But now, further, $F_1h_1 + F_2h_2 + \ldots$ is the sum of the statical moments of F_1, F_2, &c., with respect to the surface OHR of the water, and $= Fh$, F representing the area of the whole surface, and h the depth SO of its centre of gravity below the level; hence, the aggregate normal pressure against the plane surface is $P = Fh\gamma$. We mean here, by the head of water of a surface, the depth SO of its centre of gravity below the level of the water; the general rule, therefore, is true that: *the pressure of water against a plane surface is equivalent to the weight of a column of water whose base is the surface and whose height is the head of water of the surface.*

Fig. 347.



It must further be stated, that this pressure of the water is not dependent on the quantity of water which is before or below the pressed surface, that therefore, for example, a flood-gate, AC, Fig. 348, under otherwise similar circumstances, has to sustain the same pressure, whether the water to be dammed up be that of a small sluice

Fig. 348.



$ACEF$, or that of a larger dam $ACGH$, or that of a great reservoir. From the breadth $AB = CD = b$ and height $AD = BC = a$ of a rectangular flood-gate, $F = ab$, and the head of water $SO = \frac{a}{2}$; hence the pressure of water

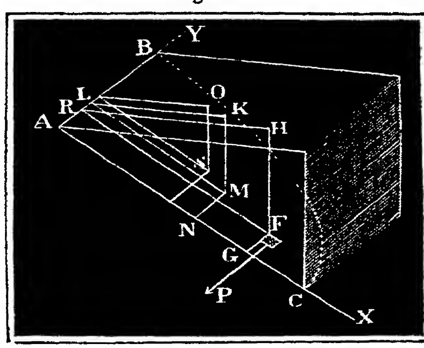
$$P = ab \,.\, \frac{a}{2}\,\gamma = \tfrac{1}{2}\, a^2\, b\gamma.$$

Therefore the pressure increases as the breadth, or as the square of the height of the pressed surface.

Example. If the water stand $3\frac{1}{2}$ feet high before a board of oak 4 feet broad, 5 feet high, and $2\frac{1}{2}$ inches thick, what will be the force required to draw it up? The volume of the board is $4 \,.\, 5 \,.\, \frac{5}{24} = \frac{25}{6}$ cubic feet. If now we take the density of oak saturated with water from § 58 at $62{,}5 \times 1{,}11 = 67{,}3$ lbs., the weight of this board will be: $G = \frac{25}{6} \,.\, 67{,}3 = 280{,}5$ lbs. The pressure of the water against the board, and also the pressure of this last against the guides will be:
$P = \frac{1}{2} \,.\, \left(\frac{7}{2}\right)^2 \,.\, 4 \,.\, 62{,}5 = 49 \,.\, 30{,}25 = 1531{,}25$ lbs.; if now we take the co-efficient of friction for wet wood from § 161, $f = 0{,}68$, the friction of this board against its guides will be $F = f\,P = 0{,}68 \,.\, 1531{,}25 = 1041{,}25$ lbs. If to this be added the weight of the board, we shall obtain the force required to pull it up $= 1041{,}25 + 67{,}3 = 1108{,}55$ lbs.

§ 277. *Centre of Pressure.*—The resultant $P = Fh\gamma$ of the collective elementary pressures $F_1h_1\gamma$, $F_2h_2\gamma$, &c., has, like every other system of parallel forces, a definite point of application, which is called the *centre of pressure.* Equilibrium will subsist for the whole pressure of the surface, if this point be supported. The statical moments of the elementary pressures $F_1h_1\gamma$, $F_2h_2\gamma$, &c., with respect to the plane of the level OHR, Fig. 349, are: $F_1h_1\gamma \,.\, h_1 = F_1h_1^2\gamma$, $F_2h_2^2\gamma$, &c.; therefore, the statical moment of the whole pressure with respect to this plane is: $(F_1h_1^2 + F_2h_2^2 + \dots)\,\gamma$. If we put the distance KM of the centre M of this pressure from the level of the water $= z$, we shall then have the moment of pressure $= Pz = (F_1h_1 + F_2h_2 + \dots)\,z\gamma$, and by equating both moments, the depth in question of the centre M below the surface:

Fig. 349.



$$1.\; z = \frac{F_1h_1^2 + F_2h_2^2 + \dots}{F_1h_1 + F_2h_2 + \dots}, \text{ or } = \frac{F_1h_1^2 + F_2h_2^2 + \dots}{Fh},$$

if, as before, F represent the area of the whole surface, and h the depth of its centre of gravity below the surface. To determine this pressure completely we must know further its distance from another plane or line. If we put the distances F_1G_1, F_2G_2, &c., of the elements of the surface F_1F_2, &c., from the line AC which determines

the angle of inclination of the plane $= y_1 y_2$, &c., we shall then have the moments of the elementary pressures with respect to this line $= F_1 h_1 y_1 \gamma$, $F_2 h_2 y_2 \gamma$, &c., therefore, the moment of the whole surface $= (F_1 h_1 y_1 + F_2 h_2 y_2 + \ldots)\,\gamma$; and if we represent the distance MN of the centre N from this line by v, we shall then have the moment also $= (F_1 h_1 + F_2 h_2 + \ldots)\, v\gamma$; if, lastly, we make both moments equal, we shall obtain the second ordinate:

$$2.\quad v = \frac{F_1 h_1 y_1 + F_2 h_2 y_2 + \ldots}{F_1 h_1 + F_2 h_2 + \ldots}, \text{ or } = \frac{F_1 h_1 y_1 + F_2 h_2 y_2 + \ldots}{Fh}.$$

If α be the angle of inclination of the plane ABC to the horizon, and x_1, x_2, &c., the distances F_1R_1, F_2R_2, &c., of the elements F_1, F_2, &c., as likewise u the distance of the centre of pressure M from the line of intersection AB of the plane with the level of the water, we shall then have:

$$h_1 = x_1 \sin.\ \alpha,\ h_2 = x_2 \sin.\ \alpha, \text{ \&c., as well as } z = u \sin.\ \alpha;$$

and if these values be put into the expressions for z and v, we shall then obtain:

$$u = \frac{F_1 x_1^2 + F_2 x_2^2 + \ldots}{F_1 x_1 + F_2 x_2 + \ldots} = \frac{\text{moment of inertia}}{\text{statical moment}}, \text{ and}$$

$$v = \frac{F_1 x_1 y_1 + F_2 x_2 y_2 + \ldots}{F_1 x_1 + F_2 x_2 + \ldots} = \frac{\text{centrifugal moment}}{\text{statical moment}}.$$

We may, therefore, find the distances u and v of the centre of pressure from the horizontal axis AY, and from the axis AX formed by the line of fall, if we divide the statical moment of the surface with respect to the first axis, once by its moment of inertia with respect to the same axis, and a second time by its centrifugal moment with respect to both axes. The first distance is at once the distance of the centre of suspension from the line of intersection with the line of the water (§ 251). It is easy to see that the centre of pressure coincides perfectly with the centre of percussion, determined in § 270, if the line of intersection AY of the surface with the level, be regarded as the axis of revolution.

If the pressed surface is a rectangle AC, Fig. 350, with horizontal base CD, the centre of pressure M will be found in the line LK let fall upon CD bisecting the basis, and will be distant $\frac{2}{3}$ of this line from the side AB in the surface of water. If this rectangle does not

Fig. 350.

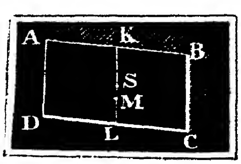


Fig. 351.

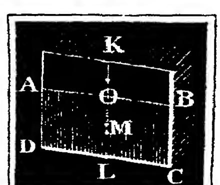
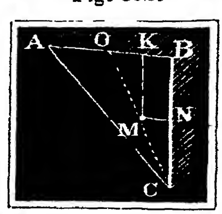


Fig. 352.



reach the surface as in Fig. 351, if further the distance KL of the lower base CD from the surface be l_1, and that of the upper base

$AB = l_2$, we then have the distance KM of the centre of pressure from the fluid surface:

$$u = \frac{2}{3} \cdot \frac{l_1^3 - l_2^3}{l_1^2 - l_2^2}.$$

For the case of a right-angled triangle ABC, Fig. 352, whose base AB lies in the fluid surface, the distance KM of the centre of pressure M from AB (§ 223), $u = \frac{\frac{1}{6} F \cdot l^2}{\frac{1}{3} F \cdot l} = \frac{1}{2} l$, if l represent the height BC of the triangle, and the distance of the same point from the other leg, as this point in every case lies in the line CO bisecting the triangle, which passes from the point O to the middle point of the base, $NM = v = \frac{1}{4} b$, where b represents the base AB.

If the point C lies in the surface, as in Fig. 353, therefore, the base AB below this point, we have

$$KM = u = \frac{\frac{1}{2} Fl^2}{\frac{2}{3} Fl} = \frac{3}{4} l \text{ and } NM = v = \frac{3}{4} \cdot \frac{b}{2} = \frac{3}{8} b.$$

Fig. 353.

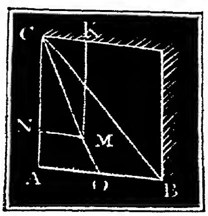
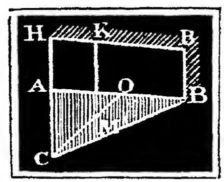


Fig. 354.



If the whole triangle ABC, Fig. 354, be under water, if the base AB is at a distance $AH = l_2$, and the point a distance $CH = l_1$ from the surface HR, we then have the distance MK from the surface HR:

$$u = \frac{\frac{1}{18} F (l_1 - l_2)^2 + F\left(l_2 + \frac{l_1 - l_2}{3}\right)^2}{F\left(l_2 + \frac{l_1 - l_2}{3}\right)}$$

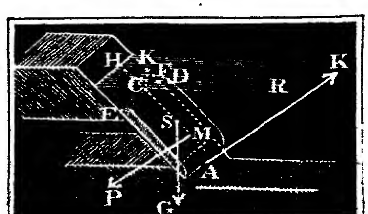
$$= \frac{\frac{1}{18}(l_1 - l_2)^2 + \frac{1}{9}(2\, l_2 + l_1)^2}{\frac{1}{3}(2\, l_2 + l_1)}$$

$$= \frac{l_1^2 + 2\, l_1\, l_2 + 3\, l_2^2}{2\,(l_1 + 2\, l_2)}.$$

In a similar manner the centres of pressure may be determined for other figures.

Example. What force K must be expended to draw up a trap-door AC turning about an axis EF, Fig. 355? Let its length $CA = 1\frac{1}{4}$ feet, its breadth $EF = 1\frac{1}{2}$ feet, its weight $= 35$ lbs.; further, the distance CK of the axis of revolution C from the surface HR, measured in the plane of the door, $= 1$ foot, and the angle of inclination of this plane to the horizon $= 68°$.

Fig. 355.



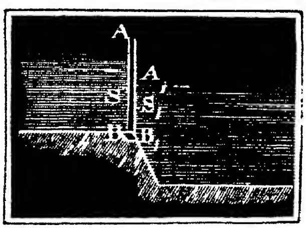
The pressed surface is $F = \frac{3}{2} \cdot \frac{5}{4} = \frac{15}{8}$ square feet, and the head of water or the depth of its centre of gravity below the surface, $h = HS\, sin.\, \alpha = (HC + CS)\, sin.\, \alpha = (HC + \frac{1}{2} AC)\, sin.\, \alpha = \left(1 + \frac{1}{2} \cdot \frac{5}{4}\right) sin.\, 68^\circ = \frac{13}{8}\, sin.\, 68^\circ = \frac{13\,.\,0{,}92718}{8} = 1{,}5067$ feet; hence, the pressure of water on the surface is. $P = Fh\gamma = \frac{15}{8}\,.\,1{,}5067\,.\,66 = 186{,}45$ lbs. The arm of this force about the axis of revolution is the distance CM of the centre of pressure M from this axis; therefore $= HM - HC$

$$= \frac{2}{3} \cdot \frac{l_1^3 - l_2^3}{l_1^2 - l_2^2} - l_2 = \frac{2}{3} \cdot \frac{\left(\frac{9}{4}\right)^3 - \left(\frac{4}{4}\right)^3}{\left(\frac{9}{4}\right)^2 - \left(\frac{4}{4}\right)^2} - 1 = \frac{1}{6} \cdot \frac{729 - 64}{81 - 16} - 1$$

$= 0{,}705$ ft.; hence the statical moment of the pressure of water $= 186{,}45\,.\,0{,}705 = 131{,}46$ ft. lbs. If the centre of gravity S of the trap-door lies about half the length $CS = \frac{1}{2} \cdot \frac{5}{4} = \frac{5}{8}$ feet from the axes of revolution, the arm CD of the weight of the revolving door will be $= CS\, cos.\, \alpha = \frac{5}{8}\,.\, cos.\, 68^\circ = \frac{5}{8}\,.\,0{,}3746 = 0{,}2341$ ft., and hence the statical moment of this weight $= 35\,.\,0{,}2341 = 8{,}19$ ft. lbs. By the addition of both moments, we obtain the whole moment for drawing up the trap-door $= 131{,}46 + 8{,}19 = 139{,}65$ ft. lbs.; and if the force K for this effect act at the arm $CA = 1{,}25$ feet, its amount will be $= \frac{139{,}65}{1{,}25} = 112$ lbs.

§ 279. If water presses against both sides of a plane surface AB, Fig. 356, there arises from the resultant forces corresponding to the two sides a new resultant, which is obtained by the subtraction of the former, because these two act oppositely to each other.

Fig. 356.



If F is the area of the pressed portion on the one side of the surface AB, and h the depth AS of its centre of gravity below the level of the water; further, F_1 the area of the portion $A_1 B_1$ on the other side of the surface, and h_1 the depth $A_1 S_1$ of its centre of gravity below the corresponding level of the water, we then have for the resultant sought, $P = Fh\gamma - F_1 h_1 \gamma = (Fh - F_1 h_1)\,\gamma$.

If the moment of inertia of the first portion of the fluid surface with respect to the line in which the plane of the surface intersects that of the water, $= Fx^2$, the statical moment of the pressure of water of the one side is, therefore, $= Fx^2\,.\,\gamma$; if, further, the moment of inertia of the second portion with respect to the line of intersection with the second surface of water $= F_1 x_1^2$, the statical moment of the pressure of water of the other side about the axis lying on the second surface is then $= F_1 x_1^2 \gamma$. Further, if the distance AA_1 of the axes $= a$, we then obtain the augmentation of the last moment in its transit from the axis A_1 to the axis A, $= F_1 h_1\, a\, \gamma$, and hence the statical moment of the pressure of water with respect to the axis in the first surface

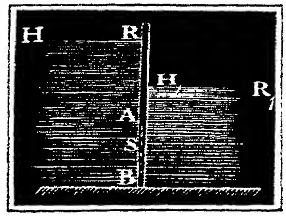
$$= F_1 x_1^2 \gamma + F_1 h_1\,.\,a\,.\,\gamma = (F_1 x_1^2 + F_1 a h_1)\,\gamma.$$

From this, then, it follows that the statical moment of the difference

of both mean pressures $= (Fx^2 - F_1x_1^2 - a\ F_1h_1)\ \gamma$, and the arm of this latter force, or the distance of the centre of pressure from the axis in the first surface of water is:

$$u = \frac{Fx^2 - F_1x_1^2 - a\ F_1h_1}{Fh - F_1h_1}$$

Fig. 357.



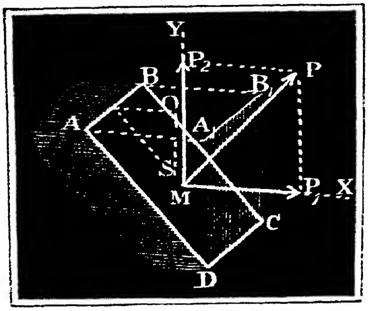
If the portions of surface pressed are equal to one another, which takes place when, as Fig. 357 represents, the entire surface AB is below the water, we have then more simply $P = F\,(h - h_1)\ \gamma$ and $u = h$; the last, because $h - h_1 = a$, and $x_1^2 = x^2 - 2\ a\ h + a^2$ (§ 217). In the last case, therefore, the pressure is equivalent to the weight of a column of water, whose base is the surface pressed, and whose height is the difference of altitude $R\,H_1$ of both surfaces of water, and the centre of pressure coincides with the centre of gravity S of the surface. This law is also further correct if both surfaces of water are besides further pressed by equal forces, for example, by a piston or by the atmosphere. For this pressure upon each unit of surface $= p$, and therefore the corresponding height of a column of water $x = \frac{p}{\gamma}$ (§ 275), we have then to substitute for h, $h + x$, and for h_1, $h_1 + x$; and by subtraction, we have the residuary force $P = (h + x - [h_1 + x])\ F\gamma = (h - h_1)\ F\gamma$. For this reason, the pressure of the atmosphere in hydrostatic investigations is generally left out of consideration.

Example. The height AB of the upper surface of water in a canal, Fig. 356, amounts to 7 feet, the water in the lock stands 4 feet high at the sluice-gate, and the breadth of the canal and of the lock measure 7,5 feet, what mean pressure has the sluice-gate to sustain? It is $F = 7 \,.\, 7{,}5 = 52{,}5$, and $F_1 = 4 \,.\, 7{,}5 = 30$ square feet. Further, $h = \frac{1}{2} \,.\, 7 = \frac{7}{2}$ and $h_1 = \frac{4}{2} = 2$ feet, $a = 7 - 4 = 3$ feet, $x^2 = \frac{1}{3} \,.\, 7^2 = \frac{49}{3}$ and $x_1^2 = \frac{1}{3} \,.\, 4^2 = \frac{16}{3}$; hence it follows, that the mean pressure sought is: $P = (Fh - F_1h_1)\ \gamma$ $= (52{,}5 \,.\, \frac{7}{2} - 30 \,.\, 2) \,.\, 62{,}5 = 123{,}75 \,.\, 62{,}5 = 7734{,}375$ lbs.; and the depth of its point of application below the surface of the water is:

$$u = \frac{52{,}5 \,.\, \frac{49}{3} - 30 \,.\, \frac{16}{3} - 3 \,.\, 60}{52{,}5 \,.\, \frac{7}{2} - 60} = \frac{517{,}5}{123{,}75} = 4{,}182 \text{ feet.}$$

§ 280. *Pressure in a Definite Direction.*—In many cases it is of importance to know only one part of the pressure acting in a definite direction upon a surface. In order to find this component, we resolve the normal pressure $MP = P$ of the surface $AC = F$, Fig. 358, in the given direction MX, and in the direction MY perpendicular to it into two component pressures $MP_1 = P_1$, and $MP_2 = P_2$. Let a be the angle PMX, which the normal in the given direction MX makes with the component, we shall then obtain for the components; $P_1 = P$

Fig. 358.

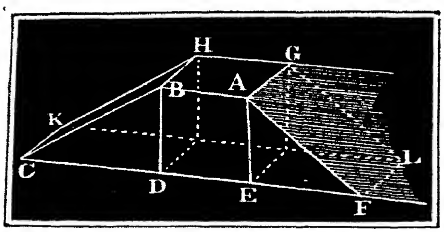


cos. α and $P_2 = P$ *sin.* α. Let a projection $A_1 B_1 CD$ of the surface AB be made on a plane at right angles to the given direction MX, we shall then have for its area F_1, the formula $F_1 = F$. *cos.* ADA_1, or since the angle of inclination ADA_1 of the surface from its projection is equal to the angle $PMX = \alpha$ between the normal pressure P and its component P_1, we then have $F_1 = F$ *cos.* α, or inversely : *cos.* $\alpha = \frac{F_1}{F}$, and hence $P_1 = P \cdot \frac{F_1}{F}$. But as the normal pressure $P = Fh\gamma$, it follows finally that $P_1 = F_1 h\gamma$, *i. e. the pressure with which water presses against a surface in a given direction, is equal to the weight of a column of water which has for base the projection of the surface perpendicular to the given direction, and for height, the depth of the centre of gravity of the surface below that of the water.*

It is important, in most cases of application, to know only the vertical or the horizontal component of the pressure of water against a surface. Since the projection at right angles to the vertical direction is the horizontal, and the projection at right angles to the horizontal direction, a vertical projection, the vertical pressure of water against a surface may be found, if the horizontal projection or its trace be considered as the surface pressed, and on the other hand the horizontal pressure of the water in any direction may be also found, if the vertical projection or the elevation of the surface at right angles to the given direction be considered as the surface pressed, but in both cases the depth of the centre of gravity of the surface below that of the water taken as the head of water.

For a prismatic dam ACH, Fig. 359, the longitudinal profile EG for the horizontal pressure of the water and the horizontal projection EL of the surface of water for the vertical pressure must be regarded as the surfaces pressed. Hence, if the length AG of the dam $= l$, the height $AE = h$, and the front slope $EF = a$, we have then the horizontal pressure of the water $= lh \cdot \frac{h}{2}\gamma = \frac{1}{2} h^2 l\gamma$, and its vertical pressure $= al \cdot \frac{h}{2}\gamma = \frac{1}{2}\, alh\gamma$. If now, further, the upper breadth of the top of the dam $= b$, the slope at the back $CD = a_1$, and the density of the mass of the dam $= \gamma_1$, we then have the

Fig. 359.



weight of the dam $= \left(b + \frac{a + a_1}{2}\right) hl\gamma_1$, and the whole vertical pressure of this against the horizontal bottom

$$= \frac{1}{2} alh\gamma + \left(b + \frac{a + a_1}{2}\right) hl\gamma_1 = \left[\frac{1}{2} a\gamma + \left(b + \frac{a + a_1}{2}\right)\gamma_1\right] hl.$$

If we put the co-efficient of friction $= f$, then the friction or force to push the dam forward is:

$$F = \left[\frac{1}{2} a\gamma + \left(b + \frac{a + a_1}{2}\right)\gamma_1\right] fhl.$$

In the case where the horizontal pressure of the water is to effect this displacement, we have:

$$\frac{1}{2} h^2 l\gamma = \left[\frac{1}{2} a\gamma + \left(b + \frac{a + a_1}{2}\right)\gamma_1\right] fhl, \text{ or more simply :}$$

$$h = f\left[a + \left(2b + a + a_1\right)\frac{\gamma_1}{\gamma}\right].$$

Therefore, in order that the dam may not be pushed away by the water, we must have:

$$h < f\left[a + \left(2b + a + a_1\right)\frac{\gamma_1}{\gamma}\right], \text{ or,}$$

$$b > \frac{1}{2}\left[\left(\frac{h}{f} - a\right)\frac{\gamma}{\gamma_1} - \left(a + a_1\right)\right].$$

For safety we assume that the base of the dam is quite permeable, on which account there is further a counter pressure from below upwards $= (b + a + a_1)\, lh\gamma$ to abstract, and we may put

$$h < f\left[\left(2b + a + a_1\right)\left(\frac{\gamma_1}{\gamma} - 1\right) - a_1\right].$$

Example. The density of the mass of a clay dam is nearly twice as great as that of water, therefore, $\frac{\gamma_1}{\gamma} = 2$ and $\frac{\gamma_1}{\gamma} - 1 = 1$; hence, for such a dam we may put simply $h > f\,(2b + a)$. According to experience, a dam will resist a long time if its height, slope and breadth at the top are equal to one another; if in the last formula we put $h = b = a$, then $f = \frac{1}{3}$, whence we must in other cases put:—

$h = \frac{1}{3}\left[(2b + a + a_1)\left(\frac{\gamma_1}{\gamma} - 1\right) - a_1\right]$, and for clay dams especially, $h = \frac{1}{3}\,(2b + a)$, and inversely, $b = \frac{3h - a}{2}$. If the height of the dam be 20 feet, and the angle of slope $\alpha = 36°$, the slope a will be

$$= h\, cotg.\, \alpha = 20 \,.\, cotg.\, 36° = 20 \,.\, 1{,}3764 = 27{,}53 \text{ feet,}$$

and hence the upper breadth of the dam $b = \frac{60 - 27{,}53}{2} = 16{,}24$ feet.

§ 281. *Pressure on Curved Surfaces.*—The law found in the last paragraph on the pressure of water in a definite direction is true only for plane surfaces, or for the separate elements of curved surfaces, but not for curved surfaces in general. The normal pressures on the separate elements of a curved surface may be resolved into lateral components parallel to a given direction, and into others acting in the plane normal to it; these components form a system of parallel forces, whose resultant gives the pressure in the given direction, and these

components may be reduced to a resultant, but the two resultants admit of no further composition when their directions do not intersect. It is not possible in general to reduce the aggregate pressures against the elements of a curved surface to a single force, but particular cases present themselves where this composition is possible.

Let G_1, G_2, G_3, &c., be the projections, and h_1, h_2, h_3, &c., the heads of water of the elements F_1, F_2, F_3, &c., of a curved surface, we then have the pressure of water in the direction perpendicular to the plane of projection:

$$P_1 = (G_1 h_1 + G_2 h_2 + G_3 h_3 + \ldots)\gamma,$$

and its moment with respect to the plane of the surface of water.

$$P_1 u = (G_1 h_1^2 + G_2 h_2^2 + G_3 h_3^2 + \ldots)\gamma.$$

If the curved surface pressed upon can be decomposed into elements which have a uniform ratio to their projections, we may then put

$$\frac{F_1}{G_1} = \frac{F_2}{G_2} = \frac{F_3}{G_3}, \text{ \&c.}, = n, \text{ we then have:}$$

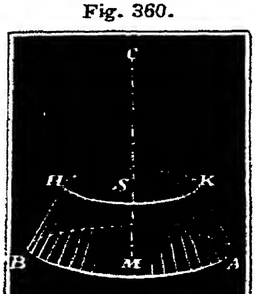
$$P_1 = \left(\frac{F_1 h_1}{n} + \frac{F_2 h_2}{n} + \ldots\right)\gamma = \left(\frac{F_1 h_1 + F_2 h_2 + \ldots}{n}\right)\gamma,$$

or, since the ratio of the entire curved surface F to its projection G, *i. e.* $\frac{F}{G}$ is $= n$,

$P_1 = \frac{Fh}{n}\gamma = Gh\gamma$; in this case we have, as for every plane surface, the pressure in any direction equivalent to the weight of a prism of water, whose basic surface is at right angles to the projection of he curved surface in the given direction, and whose height is equal to the depth of the centre of gravity of the curved surface below the surface of water.

So, for example, the vertical pressure of water against the envelope of a conical vessel ACB, filled with water, Fig. 360, is equal to the weight of a column of water which has the bottom for its base, and two-thirds of the length of the axis CM for height, because the horizontal projection of the envelope of a right cone upon its base, as likewise the envelope, may be resolved into exactly similar triangular elements, and because the centre of gravity S of the surface of the cone is distant two-thirds of the height of the cone from the vertex (§ 110). If r be the radius of the base, and h the height of the cone, we shall then have the pressure against the bottom $= \pi r^2 h\gamma$, and the vertical pressure against the envelope $= \frac{2}{3}\pi r^2 h\gamma$, but as the bottom is rigidly connected with the sides, and both pressures act opposed to each other, the force with which the vessel is pressed downwards by the water is:

Fig. 360.

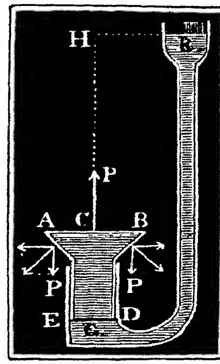


$$= \left(1 - \frac{2}{3}\right) \pi r^2 h\gamma = \frac{1}{3} \pi r^2 h\gamma =$$

to the weight of the whole mass of water. If the bottom be separated by a fine cut from the envelope, this will then press with its full force $\pi r^2 h\gamma$ downwards, or on its support, and on the other hand it would be necessary to hold down the envelope with a force $\frac{2}{3} \pi r^2 h\gamma$ to prevent its being raised off.

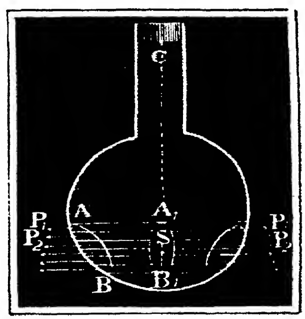
Remark. From this the force which the steam of a steam-engine or the water of a water-column machine exerts on the piston, is independent of the form of the piston. Whether the surface of pressure be augmented by being hollowed out or rounded, the pressure with which the steam or water pushes forward the piston is equivalent to the product of the cross section or horizontal projection of the piston and the pressure on a unit of surface. The pressure on the larger surface of a funnel-shaped piston AB, Fig. 361, whose greater radius $CA = CB = r$ and lesser radius $GD = GE = r_1$, is $= \pi r^2 p$, and the reaction upon the envelope $= \pi (r^2 - r_1^2) p$; hence, the residuary effective pressure is $= \pi r^2 p - \pi (r^2 - r_1^2) p = \pi r_1^2 p$ = the cross section of the cylinder multiplied by the pressure on a unit of surface.

Fig. 361.



§ 282. *Horizontal and Vertical Pressure.*—Whatever may be the form of a curved surface, AB, Fig. 362, the horizontal pressure of the water against it is always equivalent to the weight of a column of water, whose base is the vertical projection A_1B_1 of the surface perpendicular to the given direction of pressure, and whose height of pressure is the depth CS of the centre of gravity S of the projection below the surface of water. The correctness of this follows directly from the formula $P_1 = (G_1h_1 + G_2h_2 + \ldots)\gamma$, when we consider that the height of pressure h_1, h_2, &c., of the elements of the surface are also the heights of pressure of their projections, that, therefore, $G_1h_1 + G_2h_2 + \ldots$ is the statical moment of the whole projection, *i. e.* the product Gh of the vertical projection G and the depth h of its centre of gravity below the surface of water. We have here, therefore, again to put $P_1 = Gh\gamma$, and to consider h as the height of pressure of the vertical projection.

Fig. 362.

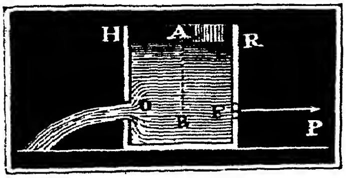


The vertical section which divides a vessel containing water into two equal or unequal parts, is at once the vertical projection of the two parts, but the horizontal pressure on one part of the wall of the vessel is proportional to the product of its vertical projection and to the depth of its centre of gravity below the surface of the water, consequently the horizontal pressure on a part of the wall of the vessel is exactly equal in amount to the oppositely acting horizontal pressure on the part opposite, and consequently the two forces balance each

other in the vessel; the whole vessel is therefore equally pressed by the enclosed water in all horizontal directions.

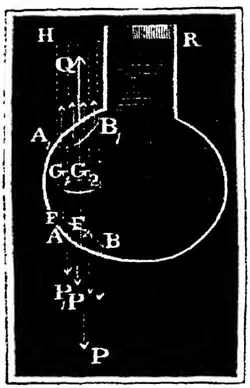
If an opening O be made in the side of a vessel HBR, Fig. 363, the part of the pressure corresponding to the section of this opening disappears, and the pressure on the oppositely situated portion of the surface F now comes into action. Whilst, therefore, the water flows out at the lateral aperture, an equal distribution of the horizontal pressure no longer takes place over the whole extent, and there ensues a reaction opposite to the motion of the flowing water: $P = Fh\gamma$, F being the projection of the aperture, and h the height of pressure of its projection. By this reaction the vessel may be set into motion.

Fig. 363.



The vertical pressure of water is $P_1 = G_1 h_1 \gamma$ against an element of surface F_1, Fig. 364, of the side of the vessel, since the horizontal projection G_1 may be regarded as the transverse section, and the height of pressure h_1 as the height, and therefore $G_1 h_1$ as the volume of a prism, equivalent to the weight of a column of water HF_1 incumbent on the element, and reaching the surface of the water. The elements of the surface which make up a finite part of the bottom, or side of the vessel, hence suffer a vertical pressure which is equivalent to the weight of all the incumbent columns of water, *i. e.* to the weight of a column of water incumbent on the whole portion. Let this volume $= V_1$, we then obtain for the vertical pressure $P = V_1\gamma$. For another portion A_1B_1, which lies vertically above the former, we have the vertical pressure opposed to it $Q = V_2\gamma$; but if both portions are rigidly connected with each other, there results from the two forces the force acting vertically downwards $R = (V_1 - V_2)\gamma = V\gamma =$ *to the weight of the columns of water contained between the two portions of the surface.* If, lastly, we apply this law to the whole vessel, it follows that *the aggregate vertical pressure of the water against the vessel is equivalent to the weight of the enclosed mass of water.*

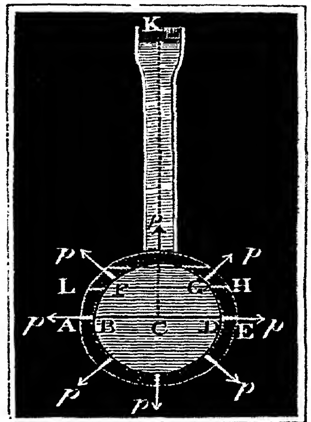
Fig. 364.



§ 283. *Thickness of Pipes.*—The application of the laws of the pressure of water to pipes, boilers, &c., is of particular importance. That these vessels may adequately resist the pressure, and be prevented bursting from its effect, we must give a certain thickness to their sides, corresponding to the head of water and the internal width. The bursting of a pipe may take place in various ways, viz., transversely or longitudinally; the latter happens more frequently than the former, as will be soon understood from what follows.

The width of the pipe $BD = 2r$, Fig. 365, and the head of water

Fig. 365.



$CK=h$, therefore the pressure on a unit of surface $p-h\gamma$, we then have the whole pressure in the direction of the axis of the pipe $=\pi r^2 p=\pi r^2 h\gamma$; if the thickness of the side $AB=DE=e$, we then have the transverse section of the mass of the pipe $=\pi\,(r+e)^2-\pi r^2=2\,\pi re+\pi e^2=2\,\pi re\left(1+\frac{e}{2r}\right)$, and if lastly we put the modulus of elasticity $=K$, we then have the pressure for rupture over the whole section of the pipe

$$=2\,\pi re\left(1+\frac{e}{2r}\right)K,$$

for this reason we have now to put:

$$2\,\pi re\left(1+\frac{e}{2r}\right)K=\pi r^2 p,$$

or approximately and more simply $2\,e\,K=rp$, and hence the thickness of the pipe $e=\frac{rp}{2K}=\frac{ph\gamma}{2K}$. In order, therefore, to avoid any transverse rent in the pipe or in the boiler, the thickness of the sides must be made $e>\frac{rp}{2K}$. Of all longitudinal rents, AE, LH, &c., those running diametrically, such as AE, take place the most easily, because they have the smallest area, whence we must only take these into account. Let us consider a portion of a pipe of the length l, and let us have regard to the occurrence of a rent of the length l, we then obtain a transverse section of the surface of separation $=le$, and hence the force for rupture in this surface leK. For two oppositely situated rents this force is $=2\,leK$, whilst the pressure of water for each half of the pipe is proportional to the transverse section $2\,rl$, and hence is $=2\,rlp$. By equating the two expressions, it follows that $2\,leK=2\,rlp$, i. e. $e\,K=rp$, therefore the thickness $e=\frac{rp}{K}$. To provide against longitudinal rents, the sides must be made as thick again, as to provide against transverse rents.

From the formula $e=\frac{rp}{K}=\frac{rh\gamma}{K}$, it follows that *the strength of similar pipes is as the widths and as the heads of water or pressures upon a unit of surface.* A pipe three times the width of another, which has five times the pressure to sustain on each unit of surface that the other has, must have its sides fifteen times as thick.

Hollow spheres, which have to sustain a pressure p from within on each unit of surface, require a thickness $e=\frac{rp}{2K}$, because here the projection of the surface of pressure is the greatest circle πr^2, and the surface of separation the ring $2\,\pi re\left(1+\frac{e}{2r}\right)$, or approximately for a smaller thickness $=2\,\pi re$.

The formula found give for $p=o$, also $e=o$, for this reason,

therefore, pipes which have no internal pressure to sustain, may be made indefinitely thin; but as each pipe must sustain a certain pressure from its own weight, we must still give to it a certain thickness e_1, to obtain the strength of a tube which will resist under all circumstances. Hence, for cylindrical pipes or boilers we must put $e = e_1 + \frac{rh\gamma}{K}$, or more simply, if d represents the interior diameter of the pipe, n the pressure in atmospheres, each corresponding to a column of water 33 ft. high, and μ a number from experiment $e = e_1 + \mu nd$.

From experiments made we must take for pipes of

Iron plate . . .	$e = 0{,}00086\, nd + 0{,}12$ inches
Cast iron . . .	$e = 0{,}00238\, nd + 0{,}33$ "
Copper	$e = 0{,}00148\, nd + 0{,}16$ "
Lead	$e = 0{,}00242\, nd + 0{,}20$ "
Zinc	$e = 0{,}00507\, nd + 0{,}16$ "
Wood	$e = 0{,}0323\, nd + 1{,}04$ "
Natural stones . .	$e = 0{,}0369\, nd + 1{,}15$ "
Artificial stones . .	$e = 0{,}0538\, nd + 1{,}53$ "

Example. If a perpendicular water-column machine has cast-iron pipes of 10 inches inner width, how thick must these be at 100, 200, and 300 feet depths? From the formula, for 100 feet pressure, this thickness is:

$$= 0{,}00238 \cdot \frac{100}{33} \cdot 10 + 0{,}33 = 0{,}07 + 0{,}33 = 0{,}40 \text{ inches};$$

for 200 feet, $= 0{,}14 + 0{,}33 = 0{,}47$ inches; and for 300 feet pressure, $= 0{,}22 + 0{,}33 = 0{,}55$ inches. Cast iron conducting pipes are commonly proved at 10 atmospheres, for which reason, $e = 0{,}0238 \cdot d + 0{,}33$ inches; therefore, for pipes of 10 inches width, the thickness $e = 0{,}24 + 0{,}33 = 0{,}57$ inches must be given.

Remarks. The thickness of the sides of steam-boilers will be considered in the Second Part. Concerning the theory of the strength of pipes. a treatise by Brix, in the "Verhandlungen des Vereins zur Beförderung des Gewerbfleiszes in Preuszen," Jahrgang, 1834, may be consulted. The technical relations and the proving of pipes are fully treated of in Hagen's "Handbuch der Wasserbaukunst," vol. i., and in Genieys' "Essai sur les moyens de conduire, &c., les eaux."

[For a view of the general principles governing the construction and strength of cylindrical steam-boilers, the editor may refer to his paper on that subject read before the Franklin Institute, July 26, 1832, and published in its Journal, in which the. relation stated in the text, between the strength required in the direction of the curvature and that in the direction of the length of the tube or boiler, was pointed out, accompanied by a table of diameters and thicknesses of boilers, with the tenacities per inch of iron required in each direction for a given pressure. See, likewise, American Journal of Science and Arts, vol. xxiii. No. 1.]

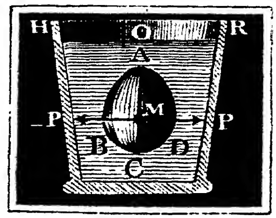
CHAPTER II.

ON THE EQUILIBRIUM OF WATER WITH OTHER BODIES.

§ 284. *Buoyancy.*—A body immersed under water is pressed upon by the water on all sides, and now the question arises as to the

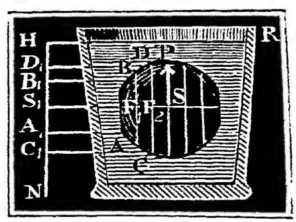
amount, direction and point of application of the resultant of all these pressures. Let us imagine this resultant to consist of a vertical and a horizontal component, and determine these forces according to the rules of § 282. The horizontal pressure of the water against a surface is equivalent to the horizontal pressure against its vertical projection, but now every projection of a body, AC, Fig. 366, is at the same time the projection of the fore part ADC and the back part ABC of its surface; hence, also, the horizontal pressure of water against the back portion of the surface of a body is equal in amount to that of the front portion, and as both pressures are exactly opposite, their resultant $= 0$. As this relation takes place for every arbitrary horizontal direction, and the vertical projection corresponding to this, it follows that the resultant of all the horizontal pressures is nothing; that, therefore, the body AC below the water is equally pressed in all horizontal directions, and for this reason exerts no effort to move forward in a horizontal direction.

Fig. 366.



To find the vertical pressure of the water against the body BCS, Fig. 367, let us suppose it made up of vertical elementary prisms, AB, CD, &c., and determine the vertical pressures on their terminating surfaces A and B, C and D. If the lenghts of these prisms are l_1, l_2, &c., the depths of their upper extremities B, D, &c., below the surface of water HR : h_1, h_2, &c., and the horizontal transverse sections F_1F_2, &c., we then have for the vertical pressures acting from above downwards against the extremities, B, D, &c., $= F_1h_1\gamma$, $F_2h_2\gamma$, &c.; on the other hand, the pressures acting from below upwards and against the extremities A, C, &c., $= F_1(h_1 + l_1)\gamma$, $F_2(h_2 + l_2)\gamma$, &c.; and it now follows, from a composition of these parallel forces, that the resultant P

Fig. 367.



$$= F_1(h_1 + l_1)\gamma + F_2(h_2 + l_2)\gamma + \ldots - F_1h_1\gamma - F_2h_2\gamma - \ldots$$
$$= (F_1l_1 + F_2l_2 + \ldots)\gamma = V\gamma,$$

if V represents the volume of the immersed body or the water displaced.

Therefore the buoyancy or the force with which the water strives to push a body immersed from below upwards, is equivalent to the weight of water displaced, or to a quantity of water which has the same volume as the submerged body.

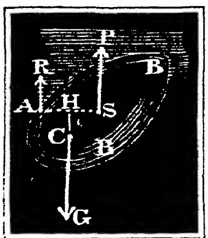
Further, to find the point of application of this resultant, let us put the distances AA_1, CC_1, &c., of the elementary columns AB, CD, &c., from a vertical plane HN : a_1, a_2, &c., and determine the moments of the forces with respect to this plane. If S is the point of application of the upward pressure, and $SS_1 = x$ its distance from that principal plane, we shall then have:

$V\gamma \cdot x = F_1\, l_1\gamma \cdot a_1 + F_2\, l_2\gamma \cdot a_2 + \dots$; and hence,

$x = \frac{F_1\, l_1\, a_1 + F_2\, l_2\, a_2 + \dots}{F_1\, l_1 + F_2\, l_2 + \dots} = \frac{V_1\, a_1 + V_2\, a_2 + \dots}{V_1 + V_2 + \dots}$, if V_1, V_2, &c., represent the contents of the elementary columns. Since (from § 100) the centre of gravity is accurately determined by the same formula, it follows *that the point of application S of the upward pressure coincides with the centre of gravity of the water displaced.*

§ 285. The weight G of the body acting in an opposite direction associates itself with the buoyancy of the body immersed or under water, and from the two there arises a resultant $R = G - V\gamma$ or $= (\varepsilon - 1)\, V\gamma$, if ε be the specific gravity of the body.

Fig. 368.

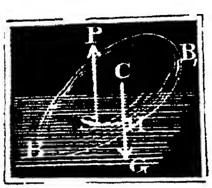


If the mass of the body be homogeneous, the centre of gravity of the displaced water will coincide with that of the body, and hence this point will be the point of application of the resultant R; but if there be not homogeneity, then these centres of gravity do not coincide, and the point of application of the resultant R deviates from both centres of gravity. Let us put the horizontal distance SH, Fig. 368, of both centres of gravity from each other, $= b$, and the horizontal distance SA of the point of application A sought from the centre of gravity S of the displaced water $= a$, we shall have the equation $Gb = Ra$, from which is given:

$$a = \frac{G\, b}{R} = \frac{G\, b}{G - P}.$$

If the immersed body be left to its own gravity, the three following cases may present themselves. Either the specific gravity of the body is equal to that of the water, or it is greater, or it is less than the specific gravity of the water. In the first case the buoyancy is equal, in the second it is less, and in the third it is greater than the weight of the water. Whilst, in the first case, equilibrium subsists between the weight and the buoyancy, the body must in the second case sink with the force $G - V\gamma = (\varepsilon - 1)\, V\gamma$, and, in the third case, rise with the force $V\gamma - G = (1 - \varepsilon)\, V\gamma$. The rising goes on only as long as the mass of water V_1, cut off from the plane of the surface and displaced by the body, has the same weight as the entire body. The weight $G = V\varepsilon\gamma$ of the body BB_1, Fig. 369, and the buoyancy $P = V_1\gamma$ now constitute a couple, by which the body is made to revolve until the directions of both coincide, or until the centre of gravity of the body lies in one and the same vertical line with the centre of gravity of the displaced water.

Fig. 369.

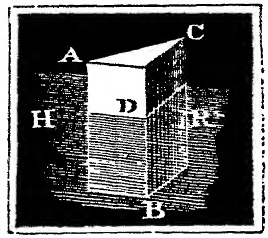


The line passing through the centre of gravity of the floating body and through that of the displaced water, is called *the axis of floatation;* and on the other hand, the section of the body formed by the plane of the surface of the water, *the plane of floatation.* Every plane which divides a body, so that one part is to the whole as the specific gravity of the body to that of the fluid, and that the centres of gravity of the

two parts lie in a line normal to this plane, is a plane of floatation of the body.

§ 286. *Depth of Floatation.* If the figure and weight of a floating body be known, the depth of immersion may be calculated beforehand, with the help of the previous rule. If G be the weight of the body, we may then put the volume of the displaced water $V = \frac{G}{\gamma}$; if we combine with it the stereometrical formula for the volume V, we shall obtain the equation of condition. Hence, for the prism ABC, Fig. 370, with vertical axis, for example, $V = Fy$, if F represent the section and y the depth BD of immersion, $Fy = \frac{G}{\gamma}$ and $y = \frac{G}{F\gamma}$. For a pyramid ABC, Fig. 371, whose vertex floats under the water, $V = \frac{1}{3} f y^3$, if f represents the section at the distance of unity from the vertex; hence it follows, that:

Fig. 370.



$$\tfrac{1}{3} f y^3 = \frac{G}{\gamma}, \text{ and hence the depth } CE = y = \sqrt[3]{\frac{3\,G}{f\,\gamma}}.$$

Fig. 371.

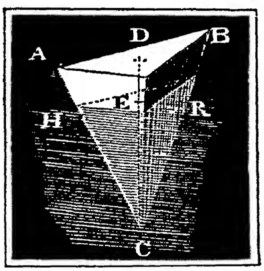
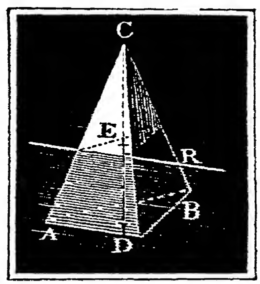


Fig. 372.



For a pyramid ABC, Fig. 372, floating with its base below the water, the distance is given $CE = y_1$ of the vertex from the surface, from the height h of the entire pyramid, if we put:

$$V = \tfrac{1}{3} f (h^3 - y_1^3),\ y_1 = \sqrt{h^3 - \frac{3\,G}{f\gamma}}.$$

For a sphere AB, Fig. 373, with the radius $CA = r$,

Fig. 373.

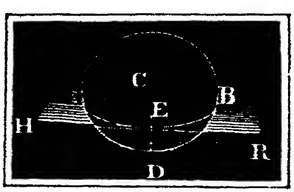
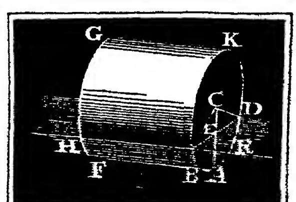


Fig. 374.



$V = \pi\, y^2 \left(r - \frac{y}{3} \right)$, hence we shall have to solve the cubic equation $y^3 - 3\, r\, y^2 + \frac{3\, G}{\pi \gamma} = 0$, to find the depth of immersion DE of the sphere.

For a floating cylinder AG, with horizontal axis, Fig. 374, of a radius $BC = DC = r$, if α° be the angle BCD subtended at the centre by the arc immersed, the depth of immersion $AE = y = r\,(1 - \cos. \frac{1}{2}\, \alpha)$, but to find the arc immersed, we must put the volume of the water displaced = to the segment $\frac{r^2\, \alpha}{2}$ less the triangle $\frac{r^2 \sin. \alpha}{2}$, multiplied by the length $GK = l$ of the cylinder; therefore, $(\alpha - \sin. \alpha) \frac{l r^2}{2} = \frac{G}{\gamma}$, and solve the equation $\alpha - \sin. \alpha = \frac{2\, G}{l r^2 \gamma}$, by approximation, with respect to α.

Examples.—1. A wooden sphere, of 10 inches diameter, floats $4\frac{1}{2}$ inches deep, the volume of water displaced by it is then:

$$V = \pi \left(\frac{9}{2}\right)^2 \left(5 - \frac{9}{6}\right) = \frac{\pi\ 81\,.\,7}{8} = \frac{567\,.\,\pi}{8} = 222{,}66 \text{ cubic inches,}$$

whilst the solid contents of the sphere are $\frac{\pi\, d^3}{6} = \frac{\pi\,.\,10^3}{6} = 523{,}6$ cubic inches.
From this, 523,6 cubic inches of the mass of the sphere weigh as much as 222,66 cubic inches of water, and it follows that the specific gravity of the former is:

$$\varepsilon = \frac{222{,}66}{523{,}6} = 0{,}425\,..$$

2. How deep will a wooden cylinder of 10 inches diameter and specific gravity $\varepsilon = 0{,}425$ sink? $\frac{\alpha - \sin.\alpha}{2} = \frac{\pi\, r^2\, l\,.\,\varepsilon\gamma}{l\, r^2\, \gamma} = \pi\, \varepsilon = 0{,}425\,.\,\pi = 1{,}3352$; now a table of segments gives for the area $\frac{\alpha - \sin.\alpha}{2} = 1{,}32766$ of a circular segment, the angle subtended at the centre by the arc $\alpha^\circ = 166^\circ$, and for $\frac{\alpha - \sin.\alpha}{2} = 1{,}34487$, the same angle $= 167^\circ$; hence, simply, the angle subtended at the centre corresponding to the slice 1,3352 is:

$$\alpha^\circ = 166^\circ + \frac{1{,}3352 - 1{,}32766}{1{,}34487 - 1{,}32766}\,.\,1^\circ = 166^\circ + \frac{754^\circ}{1721} = 166^\circ\,26'\text{; therefore the depth}$$

of immersion:

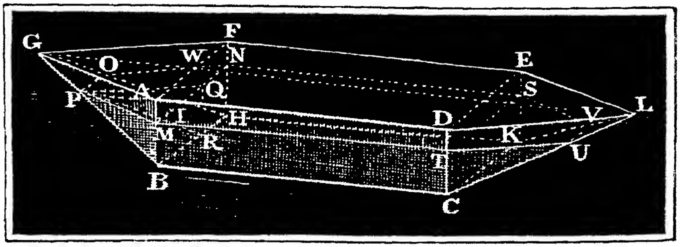
$$y = r\,(1 - \cos.\tfrac{1}{2}\,\alpha) = 5\,(1 - \cos.\,83^\circ\,13') = 5\,.\,0{,}8819 = 4{,}41 \text{ inches.}$$

§ 287. The determination of the depth of immersion occurs chiefly in the case of ships, boats, &c. If these have a regular form, the depth may be calculated from geometrical formulæ; but if this regularity fails, or the law of configuration is not known, or if the form is very complex, the depth of immersion must then be determined by experiment.

An example of the first case is in the body $ACLEG$ (a pointed scow), represented in Fig. 375, bounded by plane surfaces. It consists of a parallelopiped ACE, and of two four-sided pyramids BFG and CEL, forming the head and the stern, and its plane of floatation is composed of a parallelogram MS, and two trapeziums MO and SU, and cuts off a bulk of water, consisting of a parallelopiped MCS, and two triangular

prisms PNR, and two quadrilateral pyramids BQP. If we put the length AD of the middle portion $= l$, the breadth $AF = b$, and the

Fig. 375.



depth $AB = h$; further, the length GW of each of the two ends $= c$, and the depth of immersion, *i. e.*, $BM = CT = y$, the immersed part MCS of the middle portion will be: $= \overline{MN} \times \overline{MT} \times \overline{MB} = lby$. The base of the quadrilateral pyramid BQP is $\overline{BM}\,.\,\overline{BR}$, and the height PJ, hence the solid contents of this pyramid $= \frac{1}{3}\,\overline{BM}\,.\,\overline{BR}\,.\,\overline{PJ}$. But now:

$$BM = y,\ BR = \frac{BP}{BG}\,.\,BH = \frac{BM}{BA}\,.\,BH = \frac{y}{h}\,.\,b = \frac{by}{h},$$

and likewise:

$$PJ = \frac{BM}{BA}\,.\,GW = \frac{y}{h}\,c = \frac{cy}{h},$$

hence the contents of both pyramids are:

$$= 2\,.\,\tfrac{1}{3}\,.\,y\,.\,\frac{by}{h}\,.\,\frac{cy}{h} = \tfrac{2}{3}\,\frac{bcy^3}{h^2}.$$

The transverse section of the triangular prism

$$RNO \text{ is} = \tfrac{1}{2}\,\overline{RQ}\,.\,\overline{PJ} = \tfrac{1}{2}\,y\,.\,\frac{cy}{h} = \frac{cy^2}{2h}, \text{ and the side}$$

$$RH = QN = b - \frac{by}{h} = b\left(1 - \frac{y}{h}\right),$$

hence the solid contents of both prisms are:

$$= 2\,.\,\frac{cy^2}{2h}\,.\,b\left(1 - \frac{y}{h}\right) = \frac{bcy^2}{h}\left(1 - \frac{y}{h}\right).$$

By addition of the three volumes found, the volume of the water displaced is known:

$$V = bly + \tfrac{2}{3}\,\frac{bcy^3}{h^2} + \frac{bcy^2}{h} - \frac{bcy^3}{h^2} = \left(l + \frac{cy}{h} - \tfrac{1}{3}\,.\,\frac{cy^2}{h^2}\right)by.$$

Now the gross weight of the boat $= G$, we then have to put:

$$\left(l + \frac{cy}{h} - \tfrac{1}{3}\,.\,\frac{cy^2}{h^2}\right)by\gamma = G, \text{ or,}$$

$$y^3 - 3\,hy^2 - \frac{3\,lh^2}{c}\,.\,y + \frac{3\,h^2\,G}{bc\gamma} = 0.$$

The depth of immersion y is determined from the loading by the solution of the last cubic equation.

Examples.—1. If the length of the middle portion $l = 50$ feet, the length of each end $c = 15$ feet, the breadth $b = 12$ feet, and the depth $h = 4$ feet, with a depth of immersion $y = 2$ feet, the whole weight amounts to
$G = [50 + 15 \,.\, \frac{1}{2} - \frac{1}{3} \,.\, 15 \,.\, (\frac{2}{4})^2] \,.\, 12 \,.\, 2 \,.\, 62{,}5 = (50 + 7{,}5 - 1{,}25) \,.\, 24 \,.\, 62{,}5 = 87235$ lbs.
2. If the clear weight of the former boat amount to 50000 lbs., we shall have for the depth of immersion: $y^3 - 12y^2 - 160y + 202{,}02 = 0$. By trial, it is easily found that this equation may be answered pretty accurately by $y = 1{,}17$, whence the depth of immersion sought may be taken as great.

Remark. To know the weight of the load of a ship, a scale is attached to both sides, which is called a water-gauge. The divisions are made from experiment, while it is observed what loads correspond to definite immersions.

§ 288. *Stability.*—The floating of bodies takes place either in an upright or an oblique position; and further, with or without stability. A body, a ship, for example, floats uprightly, if one plane through the axis of symmetry is a plane of symmetry of the body; and a body floats obliquely if it is not divided by any of the planes, which may be carried through the axis of floatation into two congruent halves. A body floats with stability, if it strives to maintain its state of equilibrium (compare § 130); if, therefore, mechanical effect is to be expended to bring it out of this position, or if it returns of itself into a position of equilibrium after having been drawn out of one. On the other hand, a body floats without stability if it passes into a new position of equilibrium after having been brought out of one by a shock or blow.

If a body ABC, Fig. 376, floating at first uprightly, is brought into an inclined position, the centre of gravity S of the water displaced passes from the plane of symmetry EF, and assumes a position S_1 on the larger half immersed. The buoyancy applied at S: $P = V\gamma$, and the weight applied at the centre of gravity C of the body, viz., $G = -P$ form a couple by which (§ 90) a revolution is produced. About whatever point this revolution may take place, the point C, yielding to the weight G, will always go down, and S_1, or another point M of the vertical S_1P, obedient to the force P, will rise, therefore the plane of symmetry, or of the axis EF of the ship, will be drawn downwards at C, and upwards at M, and hence it will remain upright if M, as in the figure, lie above C, or incline itself still more

Fig. 376.

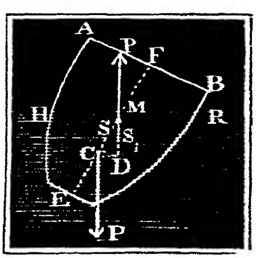
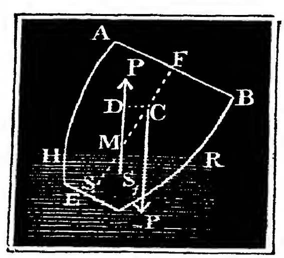


Fig. 377.



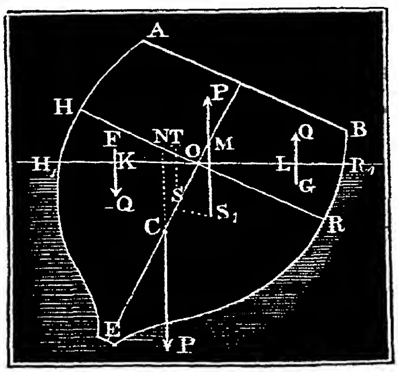
as in Fig. 377, if M lie below C. From this, then, the stability of a floating body, or ship, is dependent on the point M, in which the vertical through the centre of gravity S_1 of the displaced water intersects the plane of symmetry. This point is called the *metacentre.*

It follows, therefore, from this that a ship or other body floats with stability if its metacentre lies above the centre of gravity of the ship, and without stability if it lies below, lastly if the two points coincide, it is in a state of indifferent equilibrium.

The horizontal distance CD of the metacentre M from the centre of gravity C of the ship, is the arm of the force of the couple constituted of P, and $G = -P$, and hence the moment of the last is the measure of its stability $= P \,.\, \overline{CD}$. If we represent the distance CM by c, and the angle of revolution SMS_1 of the ship, or of the plane of its axis, by $\phi°$, we obtain for the measure of stability $S = Pc \sin \phi$; and this is, therefore, the greater, the greater the weight, the greater the distance of the metacentre from the centre of gravity of the ship, and the greater the angle of inclination of this last.

§ 289. In the last formula, $S = P\,c \sin. \phi$, the stability of the ship depends principally on the distance of the metacentre from the centre of gravity of the ship ; it is hence of importance to obtain a formula for the determination of this distance.

Fig. 378.



In the transit of the ship ABE, Fig. 378, from the upright into the inclined position, the centre of gravity S advances to S_1, the space HOH_1 is drawn out of the water, and that of ROR_1 sinks below, and the buoyancy on the one side is thereby diminished by a force Q acting at the centre of gravity F of the space HOH_1, and on the other side increased by an equal force Q applied at the centre of gravity G of the space ROR_1. Therefore, the buoyancy P applied at S_1 replaces the buoyancy P, originally applied at S, and the couple $(Q,-Q)$, or what comes to the same thing, an opposite force applied at S_1 keeps in equilibrium a force $-P$ applied at S together with a couple $(Q,-Q)$, or more simply, the couple $(P,-P)$ is in equilibrium with the couple $(Q,-Q)$. If now the transverse section $HER = H_1ER_1$ of the part of the ship immersed $= F$, and the section $HOH_1 = ROR_1$ of the space by which the ship is drawn up on the one side, and down on the other $= F_1$; if, further, the horizontal distance KL of the centre of gravity of these spaces $= a$, and that of MT of the centres of gravity S and S_1, or the horizontal projection of the space SS_1 which S describes during the rolling $= s$, we have then from the conditions of equilibrium of the two couples:

$$Fs = F_1 a, \text{ hence } s = \frac{F_1}{F} a \text{ and } SM = \frac{MT}{\sin. \phi} = \frac{s}{\sin. \phi} = \frac{F_1 a}{F \sin. \phi}.$$

The line $CM = c$, appearing as factor in the measure of the stability is $= CS + SM$; hence, if further, we represent the distance CS

of the centre of gravity C of the ship from the centre of gravity of the displaced water by e, we obtain the measure of the stability

$$S = Pc\ sin.\ \phi = P\left(\frac{F_1 a}{F} + e\ sin.\ \phi\right).$$

If the angle of revolution be small, the transverse sections HOH_1 and ROR_1 may be regarded as equally small triangles; if we represent the breadth $HR = H_1R_1$ of the ship at the place of immersion by b, we may then put

$$F_1 = \tfrac{1}{2} \cdot \tfrac{1}{2}\, b \cdot \tfrac{1}{2}\, b\, \phi = \tfrac{1}{8}\, b^2 \phi, \text{ and } KL = a = 2 \cdot \tfrac{2}{3}\, \frac{b}{2} = \tfrac{2}{3}\, b,$$

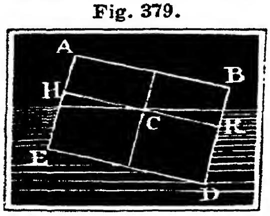
as also $sin.\ \phi = \phi$, from which the stability is:

$$S = P\left(\tfrac{1}{12}\frac{b^3 \phi}{F} + e\,\phi\right) = \left(\frac{b^3}{12\ F} + e\right) P\,\phi.$$

If the centre of gravity C of the ship coincides with the centre of gravity S of the displaced water, we then have $e = 0$, hence: $S = \frac{b^3}{12\ F} \cdot P\,\phi$, and if the centre of gravity of the ship lies below that of the displaced water, we then have e negative; hence $S = \left(\frac{b^3}{12\ F} - e\right) P\,\phi$. It follows also that the stability of a ship is nothing, if e be negative and at the same time $e = \frac{b^3}{12\ F}$.

It is seen from the results obtained, that the stability comes out greater, the broader the ship is, and the lower its centre of gravity lies.

Fig. 379.



Example. A rectangular figure AD, Fig. 379, of the breadth $AB = b$, height $AE = h$, and depth of immersion $EH = y$, $F = by$, and $e = -\frac{h-y}{2}$; hence, the amount of stability is: $S = P\phi\left(\frac{b^3}{12\ by} - \frac{h}{2} + \frac{y}{2}\right)$, or if the specific gravity of the mass of the body be put $= \varepsilon$,

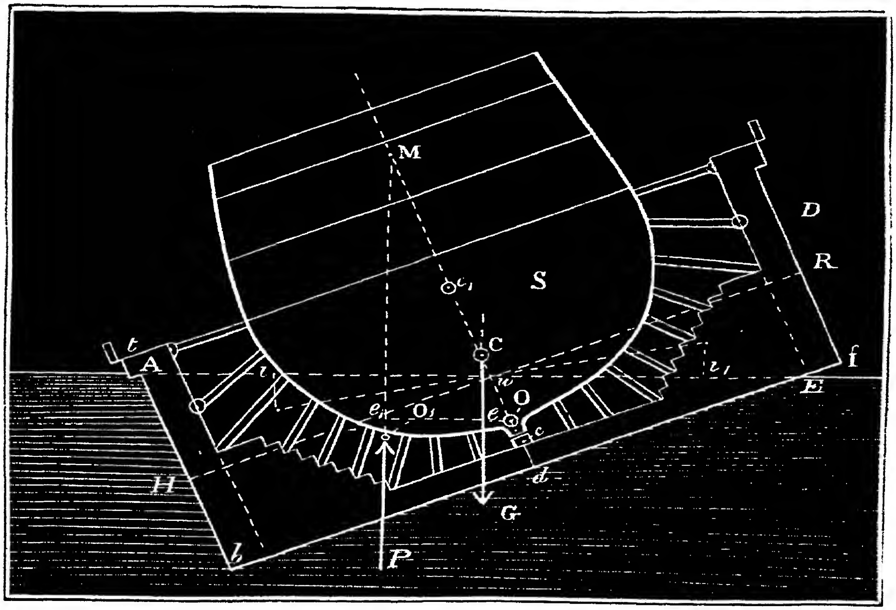
$$S = P\,\phi\left(\frac{b^2}{12\ h\varepsilon} - \frac{h}{2}\,(1 - \varepsilon)\right).$$

Hence, the stability ceases if $b^2 = 6\ h^2\, \varepsilon\, (1 - \varepsilon)$, *i. e.*, if $\frac{b}{h} = \sqrt{6\,\varepsilon\,(1 - \varepsilon)}$. For $\varepsilon = \frac{1}{2}$, $\frac{b}{h} = \sqrt{\frac{6}{2} \cdot \frac{1}{2}} = \sqrt{\frac{3}{2}} = 1{,}225$; if, therefore, the breadth is not 1,225 of the height, the body will float without any stability.

[The principles explained in this section apply not only to the construction and use of vessels of every description, and to the ballasting and lading of ships themselves, but likewise to the loading of floating docks with vessels, including their cargoes or armaments. The form of the floating mass, and the position of its centre of gravity, together with its absolute weight, must be taken into account, as well as the density of the liquid in which it floats. Thus, a floating dock D, Fig. 379_1, in the form of a rectangular prism, capable of being closed at the ends after having received the ship S, and of being freed from water, will be subject to exact calculation, if its weight and centre

of gravity be known, together with the weight of the vessel and its centre of gravity. *Example.*—Admitting that D is 90 feet wide, 36 feet

Fig. 379_1.



high from b to t, and 250 feet in length—that its weight, including ballast, is 3860 tons, and that its centre of gravity c is four feet above the bottom d; also that the ship S, weighing 5200 tons, has been received and securely shoared in place, having its centre of gravity c_1 25 feet vertically above the bottom of the keel, supposed to be 1 foot above c, then the common centre of gravity C is $\frac{26 \times 5200}{9060} = 14{,}92$ ft. above the point c, and 18,92 feet above the bottom of the dock. The total displacement in sea water (64 lbs. per cubic foot) will be 337500 cubic feet; and, consequently, in a state of repose, it will sink to the depth of $bH = \frac{337500}{250{,}90} = 15$ feet. HR will be the "load line," and the common centre of gravity of ship and dock C will be 3,92 feet above it. Should any force acting in a horizontal direction careen the dock so as to make the angle $AwH = \phi = 22° 25'$, depressing the side AH, 18 feet, what force must be applied 25 above the water line to keep the dock in this position, *i. e.* what is now the *restoring power*?

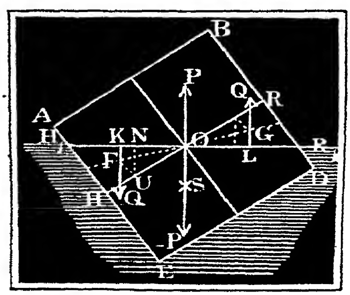
The centre of gravity of the original *quadrangular* prism of displacement $HRfb$ is at O, that of the new *triangular* prism EAb is at O_1. The weight of ship and dock C acting at the point C is 9060 tons, which is also the force P of the prism of water acting upward at

O_1—tending to restore the position of the dock. The original centre of displacement O is $18,92 - 7,5 = 11,42$ feet below the centre of gravity of ship and dock.

The area of the immersed section which is *transferred* by the careening from one side to the other is 401,5 sq. feet, and the distance transferred $ii_1 = 58,5$ feet, hence $\frac{401,5 \cdot 58,5}{90 \cdot 15} = 17,4 = e_1O =$ the horizontal distance, the centre of the whole displacement has been removed by the inclination supposed; and *sin.* $22° 25' : 17,4 =$ *rad.* : 45,63 ft. = height of the metacentre above the original centre of buoyancy O. Again, putting $r_1 = 11,42$, we have $r_1 : \sin. \phi = 11,42 : 4,35$ feet; hence, ee_1, the "equilibrating lever," or distance apart of CG and PO_1 is $17,4 - 4,35 = 13,05$ feet, and the statical moment is, therefore, $9060 \cdot 13,05 = 116058,6$ ft.–tons; which, for a distance of 25 feet from the centre of oscillation, C gives a stability or restoring power of 4642 tons.]

§ 290. *Oblique Floatation.*—The formula $S = P\left(\frac{F_1 a}{F} \pm e \sin. \phi\right)$ for the stability of a floating body may be also applied to find the different positions of floating bodies, for if we put $S = 0$ we obtain the equation for a second position of equilibrium, whose solution leads to the determination of the corresponding angle of inclination. The equation, therefore, $\frac{F_1 a}{F} \pm e \sin. \phi = 0$, must be solved with respect to ϕ.

Fig. 380.



The transverse section of a parallelopiped AD, Fig. 380, is $F = HRDE = H_1R_1DE = by$, if b be the breadth $AB = HR$, and y the perpendicular depth $EH = DR$; further, the transverse section $F_1 = HOH_1 = ROR_1$ as a rectangular triangle with the leg $OH = OR = \frac{1}{2} b$, and the leg:

$$HH_1 = RR_1 = \tfrac{1}{2} b \tan. \phi,\ F_1 = \tfrac{1}{8} b^2 \tan. \phi.$$

If, further, the centre of gravity F is distant from the base $FU = \frac{1}{3} HH_1 = \frac{1}{6} b \tan. \phi$, and if from O about $OU = \frac{2}{3} OH = \frac{1}{3} b$, it follows that the horizontal distance of the centre of gravity F from the middle $O, = OK = ON + NK = OU \cos. \phi + FU \sin. \phi = \frac{1}{3} b \cos. \phi + \frac{1}{6} b \tan. \phi \sin. \phi$, and the arm:

$$a = KL = 2\ OK = \tfrac{2}{3} b \cos. \phi + \tfrac{1}{3} b \frac{\sin. \phi^2}{\cos. \phi}.$$

According to this the equation for the oblique position of equilibrium is:

$$\frac{\frac{1}{8} b^2 \tan. \phi \left(\frac{2}{3} b \cos. \phi^2 + \frac{1}{3} b \sin. \phi^2\right)}{by \cos. \phi} - e \sin. \phi = 0,$$

or, $\frac{\sin. \phi}{\cos. \phi} = \tan. \phi$ being substituted,

$$sin.\ \phi\ [(\tfrac{1}{12} + \tfrac{1}{24}\ tang.\ \phi^2)\ b^2 - ey] = 0;$$

which equation will be satisfied by:

$$sin.\ \phi = 0 \text{ and by } tang.\ \phi = \sqrt{2}\ \sqrt{\frac{12\,e\,y}{b^2} - 1}.$$

The first equation, when $\phi = 0$, corresponds to upright, and the second to oblique floatation. The possibility of the latter requires that $\frac{ey}{b^2} > \frac{1}{12}$. If now h be the height of the parallelopiped, and ε its specific gravity, we then have:

$$y = \varepsilon\,h \text{ and } e = \frac{h - y}{2} = (1 - \varepsilon)\,\frac{h}{2}, \text{ hence it follows that}$$

$$tang.\ \phi = \sqrt{2}\ \sqrt{\frac{6\,\varepsilon\,(1 - \varepsilon)\,h^2}{b^2} - 1},$$

and the equation of condition of oblique floatation is:

$$\frac{h}{b} > \sqrt{\frac{1}{6\,\varepsilon\,(1 - \varepsilon)}}.$$

Examples.—1. If the floating parallelopiped is as high as it is broad, and has a specific gravity $\varepsilon = \frac{1}{2}$, then the $tang.\ \phi$ is $= \sqrt{2}\ \sqrt{3\,.\,\frac{1}{2} - 1} = \sqrt{3 - 2} = 1$; hence, $\phi = 45°$.

2. If the height $h = 0{,}9$ of the breadth b, and the specific gravity $\frac{1}{2}$, we have then $tang.\ \phi = \sqrt{3\,.\,0{,}81 - 2} = \sqrt{0{,}43} = 0{,}6557$; hence, $\phi = 33°\ 15'$.

§ 291. *Specific Gravity.*—The law of buoyancy of water may be applied to the determination of the density, or the specific gravity of bodies. From § 284, the upward pressure of water is equal to the weight of liquid displaced; hence if V is the volume of a body and γ_1 the density of the liquid, we then have the buoyancy $P = V\gamma_1$. If now γ_2 be the density of the mass of the bodies, we then have the weight of the body $G = V\gamma_2$; hence the ratio of the densities $\frac{\gamma_2}{\gamma_1} = \frac{G}{P}$, i. e. *the density of the body immersed is to the density of the fluid as the absolute weight of the body to the buoyancy or loss of weight by immersion.*

Therefore, $\gamma_2 = \frac{G}{P}\,\gamma_1$, and $\gamma_1 = \frac{P}{G}\,\gamma_2$; or if γ be the density of water, ε_1 the specific gravity of the liquid, and ε_2 that of the body, then will $\gamma_1 = \varepsilon_1\gamma$, and $\gamma_2 = \varepsilon_2\gamma$, $\varepsilon_2 = \frac{G}{P}\,\varepsilon_1$, and $\varepsilon_1 = \frac{P}{G}\,\varepsilon_2$. If, therefore, the weight of a body or its loss of weight by immersion is known, then the density or the specific gravity of the mass of a body may be found from the density or specific gravity of the liquid, and inversely, the density or specific gravity of the first, from the density or specific gravity of the last.

If the fluid in which the solid body is weighed is water, we then have $\varepsilon_1 = 1$, and $\gamma_1 = \gamma = 1000$ kilogrammes, or 62,5 lbs., according as we take the cubic metre or cubic foot for unit of volume, hence for this case the density of the body is:

$$\gamma_2 = \frac{G}{P}\gamma = \frac{\text{absolute weight}}{\text{loss of weight}} \text{ times the density of water,}$$

and the specific gravity:

$$\varepsilon_2 = \frac{G}{P} = \frac{\text{absolute weight}}{\text{loss of weight}}.$$

To estimate the buoyancy or loss of weight, as well as to determine the weight C, we make use of an ordinary balance, only that below one of the scale pans of this balance there is appended a hook, to which the body may be suspended by a fine thread or fine wire, whilst it dips into the water contained in a vessel underneath. A balance arranged for the weighing of bodies in water is commonly called a *hydrostatic balance.*

If the body whose specific gravity we wish to determine is lighter than water, we may connect it mechanically with another heavy body, so as to make it sink. If this heavy body loses the weight P_2, and the system the weight P_1, the loss of weight of the lighter body: $P = P_1 - P_2$, now if G represents the loss of weight of the lighter body, we have then its specific gravity:

$$\varepsilon_2 = \frac{G}{P} = \frac{G}{P_1 - P_2}.$$

If the specific gravity of a mechanical combination, or a composition of two bodies, and the specific gravities of their constituents ε_1 and ε_2 are known, from the weight of the whole, the weights G_1 and G_2 may be estimated. In every case $G_1 + G_2 = G$, and also the volume $\frac{G_1}{\varepsilon_1\gamma}$ + volume $\frac{G_2}{\varepsilon_2\gamma}$ = volume $\frac{G}{\varepsilon\gamma}$, therefore:

$\frac{G_1}{\varepsilon_1} + \frac{G_2}{\varepsilon_2} = \frac{G}{\varepsilon}$. By combining these equations we have:

$$G_1 = G\left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon_2}\right) \div \left(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}\right), \text{ and}$$

$$G_2 = G\left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon_1}\right) \div \left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right).$$

Examples.—1. If a piece of limestone, weighing 310 grains, becomes 121,5 grains lighter when under water, its specific gravity is $\varepsilon = \frac{310}{121,5} = 2,55$.—2. To find the specific gravity of a piece of oak, round which a piece of lead has been wrapped, and which has lost by being weighed in water 10,5 grains; if now the wood itself weighed 426,5 grains, and the system under water was 484,5 grains lighter than in the air, the specific gravity of the mass of wood would be:

$$\varepsilon = \frac{426,5}{484,5 - 10,5} = \frac{426,5}{474} = 0,9.$$

3. An iron vessel, completely filled with quicksilver and perfectly closed, has a net weight of 500 lbs., and has lost 40 lbs. in the water; if now the specific gravity of cast-iron = 7,2, and that of quicksilver is 13,6, the weight of the empty vessel is:

$$G_1 = 500\left(\frac{40}{500} - \frac{1}{13,6}\right) \div \left(\frac{1}{7,2} - \frac{1}{13,6}\right)$$
$$= 500\,(0,08 - 0,07353) \div (0,1388 - 0,0735)$$
$$= \frac{500 \,.\, 0,00647}{0,0653} = \frac{3235}{65,3} = 49,5 \text{ lbs.,}$$

and the weight of the enclosed quicksilver:

$$G_2 = 500\,(0{,}08 - 0{,}1388) : (0{,}07353 - 0{,}1388) = \frac{500 \,.\, 0.0588}{0{,}0653}$$
$$= \frac{2940}{6{,}53} = 450{,}2 \text{ lbs.}$$

Remark 1. For the determination of the specific gravities of liquids, meal, corn, &c., the mere weighing in open air is sufficient, because we may give to the bodies any volume at will, by filling vessels with them. If an empty bottle weighs $= G$, and the same filled with water G_1, and the weight G_2 if it contain any other substance, we shall then have the specific gravities of masses of these : $\varepsilon = \frac{G_2 - G}{G_1 - G}$. For example, to find the specific gravity of rye (not rye grains), a bottle is filled with the grains, and after much shaking, then weighed. After deduction of the weight of the empty bottle, the weight of the rye was $= 120{,}75$ grms., and the weight of an equal quantity of water $= 155{,}65$; the weight of the rye is accordingly $= \frac{120{,}75}{155{,}65} = 0{,}776$; and, therefore, 1 cubic foot of this grain weighs

$$= 0{,}776 \,.\, 62{,}5 = 48{,}5 \text{ lbs.}$$

Remark 2. The problem solved by Archimedes of finding the ratio of the constituents from the specific gravity of a mixture, and from the specific gravity of its constituents, admits only of a limited application to chemical combinations, metallic alloys, &c., because a contraction or expansion of the mass generally takes place, so that the volume of the mixture is no longer equal to the sum of the volumes of the constituents.

Remark 3. The further extension of this subject, namely, its application to the measurement of volume, &c., belongs to physics and chemistry.

§ 292. *Areometer.*—Areometers are principally used to determine the density of liquids. These instruments are hollow bodies, formed about a symmetrical axis, whose centres of gravity lie very low, and by floating perpendicularly in liquids, give their density. They are made of glass, brass, &c., and are called, according to the various purposes for which they are intended, hydrostatic balances, salzometers, hydrometers, alcoholometers, &c. There are two kinds of hydrometers, viz., the weight and the scale hydrometer. The first are often used for the determination of the weights, as was the specific gravity of solid bodies.

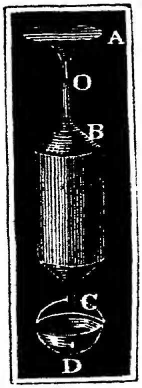
Fig. 381.

1. If V be the volume of the portion of a hydrometer ABC, Fig. 381, floating freely, and immersed up to a certain mark O in the water, G the weight of the whole balance, P the weight placed upon the plate while floating in the water, whose density may be $= \gamma$, and P_1 the weight required to be put on to make it float in any other liquid of the density γ_1, we shall then have

$$V\gamma = P + G \text{ and } V\gamma_1 = P_1 + G\text{; hence,}$$
$$\frac{\gamma_1}{\gamma} = \frac{P_1 + G}{P + G}.$$

2. If P be the weight which must be put upon the plate to make the hydrometer ABC, Fig. 382, sink up to a mark O, and P_1 the weight which must be put upon A, together with the body to be weighed, to obtain the same immersion, we shall then have simply the weight of this body $G_1 = P - P_1$. But if P_1 must be augmented by P_2, when the body to be weighed is put into the cup D under the

Fig. 382.



surface, to preserve the depth of immersion unchanged, P_2 will then be the buoyancy, and hence the specific gravity of the body:

$$\varepsilon = \frac{G_1}{P_2} = \frac{P - P_1}{P_2}.$$

Those hydrometers which have a cup suspended below for the determination of the specific gravities of solid bodies, minerals for instance, are called Nicholson's hydrometers.

Fig. 383.

3. Let the weight of a hydrometer ABC, Fig. 383, $= G$, and the volume immersed, if this balance floats in water, $= V$, then $G = V\gamma$. If the balance rise by $OX = x$, when immersed in a heavier liquid, for the transverse section F of the stem, the volume immersed is $= V - FX$, and hence $G = (V - Fx)\,\gamma_1$; the two formulæ, divided by one another, give the density of the liquid:

$$\gamma_1 = \frac{V}{V - Fx} \cdot \gamma = \gamma \div \left(1 - \frac{F}{V}x\right) = \gamma \div (1 - \mu x).$$

If the liquid in which the hydrometer is immersed, be lighter than the water, it will sink in it to a depth x, for which reason, $G = (V + Fx)\,\gamma$, and hence we must put $\gamma_1 = \gamma \div (1 + \mu x)$.

To find the co-efficient $\mu = \frac{F}{V}$, the balance is loaded with a weight P of quicksilver, which is poured in and takes the lowest position, so that while floating in water, a considerable length l of the stem to which the scale is applied, sinks lower down. If now we put $P = Fl\gamma$, we shall then obtain:

$$\mu = \frac{F}{V} = \frac{P}{Vl\gamma} = \frac{P}{Gl}.$$

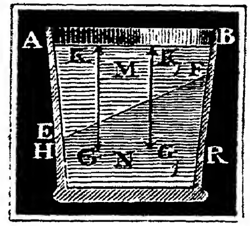
Examples.—1. If a Nicholson's hydrometer weighs 65 grains, 13,5 grains must be taken off the plate, that it may sink to the same depth in alcohol as it does in water; the specific gravity of alcohol is $= \frac{65 - 13.5}{65} = 1 - 0{,}208 = 0{,}792$.—2. The normal weight of a Nicholson's balance is 1500 grains, *i. e.* 1500 grains require to be put on to make the instrument sink to 0; from this 1030 grains must be taken by the weighing of a piece of brass placed upon the upper plate, and 121,5 to be added if this body is placed on the lower plate. The absolute weight of this piece of brass is therefore $= 1030$ grains, and its specific gravity $= \frac{1030}{121{,}5} = 8{,}47$.—3. A scale areometer, of 1162 grains weight, after having been lightened by 465 grains, rises 6 inches, and has therefore the co-efficient $\mu = \frac{465}{1162.6} = \frac{465}{6772} = 0{,}00686$. After complete filling and restoration of the weight of 1162 grains, it ascends, when floating in a saline solution, $2\frac{5}{12}$ inches; hence, the specific gravity of this is:

$$= 1 \div \left(1 - 0{,}00686 \times \frac{29}{12}\right) = 1 \div 0{,}983 = 1{,}02.$$

Remark. The further extension of this subject belongs to physics, chemistry, and technology.

§ 293. *Liquids of Different Densities.*—If several liquids, of different densities, are in the same vessel, without their exerting any chemical action upon each other, from the ready displacement of their particles, they arrange themselves above each other, according to their specific gravities, viz. the densest below, then the less dense, and then the lightest. The limiting surfaces are also in a state of equilibrium, as likewise the free surface horizontal; for as long as the surface of limitation EF between the masses M and N, Fig. 384, is inclined, columns of fluid, of different densities, like GK, G_1K_1, rest on the horizontal stratum HR, and hence the pressure on this stratum will not be everywhere the same; and lastly, no equilibrium will subsist.

Fig. 384.



In communicating tubes AB and CD, Fig. 385, the liquids arrange themselves one above the other, according to their densities, only their surfaces A and D do not lie in one and the same level. If F be the area HR of the transverse section of a piston, Fig. 386, in the one branch AB of two communicating tubes, and the height of pressure or

Fig. 385.

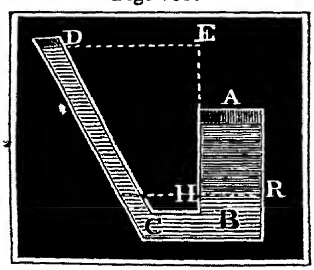
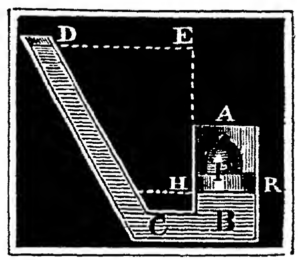


Fig. 386.



the height EH of the surface of the water in the second tube CD above HR, $= h$, we then have the pressure against the surface of the piston $P = F h\gamma$. On the other hand, if we replace the pressure of the piston by a column of liquid AH, Fig. 386, of the height $AH = h_1$ and the density γ_1, we then have $P = Fh_1 \gamma_1$; and equating both expressions, we obtain the equation $h_1 \gamma_1 = h \gamma$ or the proportion $\frac{h_1}{h} = \frac{\gamma}{\gamma_1}$.

Therefore, the heights of pressure in communicating tubes, for the subsistence of equilibrium between two different liquids, or the heights of the columns of liquid measured from the common plane of contact, are inversely as the densities or specific gravities of these liquids.

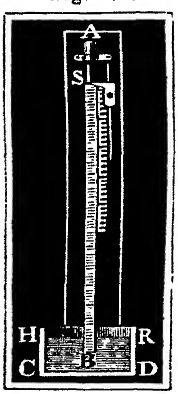
As mercury is of about 13,6 times the density of water, a column of mercury, in communicating tubes, will hold in equilibrium a column of water of 13,6 times the height.

CHAPTER III.

ON THE EQUILIBRIUM AND PRESSURE OF AIR.

§ 294. *Tension of Gases.*—The atmospheric air which surrounds us, as well as all kinds of air or gases, possesses, in virtue of the repulsive force of its parts or molecules, a tendency to occupy a greater and greater space; hence, we can obtain a limited mass of air only by confining it in perfectly closed vessels. The force with which gases endeavor to dilate themselves is called their *elasticity, tension,* or *expansive force.* It exhibits itself by pressure against the sides of the vessels which enclose it, and so far differs from the elasticity of solids and liquids, that it manifests its action in every condition of density, while the elasticity of the last-mentioned bodies in a certain state of expansion, is nothing. The pressure or tension of air and other gases is measured by the *barometer*, the *manometer*, and the *valve.* The barometer is chiefly used for determining the pressure of the atmosphere. The common, or as it is called, the *cistern barometer*, Fig. 387, consists of a glass tube, closed at one end *A* and open at the other *B*, which, when filled with mercury, is inverted, and its open end immersed in a cistern likewise containing mercury. By the inversion of this instrument, there remains in the tube a column of mercury *BS*, which (§ 393) is sustained in equilibrium by the pressure of the air on the surface of mercury *HR*. The space *AS* above the mercurial column is deprived of air, or a vacuum; hence, there is no pressure on this column from above, for which reason, the height of the mercurial column above the surface of mercury *HR* in the cistern, serves for a measure of the air's pressure.

Fig. 387.



To measure this height with precision and convenience, an accurately divided scale is appended, which runs lengthwise along the tube. A more particular description of the different barometers, and an explanation of their uses, &c., belong to the department of physics.

§ 295. It has been found by the barometer, that for a certain mean state of the atmosphere, and at places very little above the level of the sea, the air's pressure is held in equilibrium by a column of mercury, 76 centimetres, or about 28 Paris inches = 29 Prussian inches = 30 English inches nearly, (29,994 exactly.) As the specific gravity of mercury is nearly 13,6 (13,598), it follows that the pressure of the air is equivalent to the weight of a column of water,

$0{,}76 \,.\, 13{,}6 = 10{,}336$ metres $= 31{,}73$ Paris feet $= 32{,}84$ Prussian feet $= \frac{13{,}598 \,.\, 29{,}998}{12} = 33{,}988$ English feet.

The tension of the air is very often measured by the pressure it exerts upon a unit of surface. Since a cubic centimetre of mercury weighs 0,0136 kilogrammes, the pressure of the atmosphere, or the weight of a column of mercury 76 centimetres high on a base of 1 centimetre square, $= 0{,}0136 \,.\, 76 = 1{,}0336$ kilogrammes, and since a cubic inch of mercury weighs $\frac{66 \,.\, 13{,}6}{1728} = 0{,}5194$ Prussian lbs., or $\frac{62{,}5 \,.\, 13{,}6}{1728} = 0{,}491$ lbs. English, the mean pressure of the atmosphere is then $= 29 \,.\, 0{,}5194 = 15{,}05$ Prussian lbs. on the square inch, $= 2167$ lbs. on the square foot, and in English measure $= 30 \,.\, 0{,}491 = 14{,}73$ on the square inch, $= 2131{,}12$ lbs. avd. on the square foot. 14,76 lbs. per square inch is the standard usually adopted.

In mechanics, the mean pressure of the atmosphere is commonly taken as unity, and other expansive forces referred to this and assigned in atmospheric pressures, or *atmospheres*. Hence, to a pressure of n atmospheres corresponds a mercurial column of $30 \,.\, n$ inches, or a weight of 14,73 lbs. on each square inch; and inversely, to a mercurial column of h inches corresponds a tension of $\frac{h}{28} = 0{,}03571\,h$ or $= \frac{h}{30} = 0{,}0333\,h$ atmospheres, and to a pressure of p lbs. on the square inch, a tension of

$$\frac{h}{15{,}05} = 0{,}0644\,p \text{ or } \frac{h}{14{,}72} = 0{,}0678 \text{ atmospheres.}$$

The equation $\frac{h}{28} = \frac{p}{15{,}05}$ or $\frac{h}{30} = \frac{p}{14{,}73}$ give the formulæ of reduction $h = 1{,}8604\,p$ inches and $p = 0{,}5375\,h$ lbs., or $h = 2{,}036\,.\,p$ inches, and $p = 0{,}491\,h$ lbs. English. For a tension h inches $= p$ lbs., the pressure against a plane surface of F square inches: $P = Fp = 0{,}491\,Fh$ lbs. English, or $= 0{,}5375\,Fh$ lbs. Prussian.

Examples.—1. If the water in a water-pressure engine stands 250 feet above the surface of the piston, the pressure against the surface will then be $= \frac{250}{33{,}988} = 7.35$ atmospheres.—2. If the blast of a cylindrical bellows has a tension of 1,2 atmospheres, its pressure on every square inch $= 1{,}2 \,.\, 14{,}73 = 17{,}676$ lbs., and on the surface of the piston of 50 inches diameter $= \frac{\pi \,.\, 50^2}{4} \,.\, 17{,}676 = 34707$ lbs. As the atmosphere exerts a counter-pressure $\frac{\pi \,.\, 50^2}{4} \,.\, 14{,}73 = 28922{,}3$ lbs., the pressure on the piston is $= 34695 - 28922{,}3 = 5784{,}7$ lbs.

§ 296. *Manometer.*—To find the tension of gases or vapors enclosed in vessels, instruments similar to the barometer are made use of, which are called *manometers*. These instruments are filled with mercury or water, and are either open or closed; but in the latter case,

the upper part is either a vacuum or full of air. The vacuum manometer, Fig. 388, differs little from the ordinary barometer. To measure by this instrument the tension of air in a reservoir, a tube *GK* is fitted in, one end of which *G* passes into the reservoir, and the other *K* projects above the surface of mercury *CE* in the cistern of the instrument. The space *EFHR* above the mercury is hereby put into communication with the air-holder, and the air in it assumes the tension of the air in the holder, and forces into the tube a column of mercury *OS*, which sustains in equilibrium the pressure of the air which is to be measured.

The siphon manometer, *ABC*, Fig. 389, open above, gives the

Fig. 388.

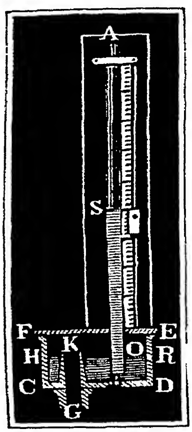


Fig. 389.

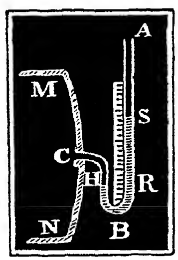
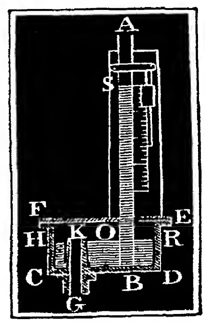


Fig. 390.

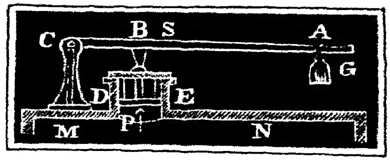


excess of tension above the pressure of the atmosphere in the vessel *MN*, because the pressure of the atmosphere on *S*, joined to that of the mercurial column *RS*, is in equilibrium with the tension. If b be the height of the barometer, and h that of the manometer, or the difference of heights *RS* of the surfaces of mercury in both branches of the manometer, we shall then have the tension of the air communicating with the shorter branch measured by the height of a column of mercury: $b_1 = b + h$, or the pressure measured on a square inch $p = 0{,}491\ (b + h)$ lbs.; or if b be the mean height of the barometer, $p = 14{,}73 + 0{,}491\ h$ lbs.

Cistern manometers, Fig. 390, *ABCE*, are more common than siphon manometers. As the air here acts through a greater quantity of mercury or water, as it may be, upon the column of fluid, its oscillations do not so quickly affect the column of fluid, and its measurement, when thus at rest, is rendered both easier and more accurate. For the sake of convenience of measuring by, or reading off from the scale, a float is not unfrequently attached to it, which rests on the mercury, and is connected with an index hand, accompanying the scale by means of a thread passing over a small roller.

The expansive force of a gas or vapor enclosed in MN may be likewise determined, but with less accuracy, by the help of a valve DE, Fig. 391, if the sliding weight is so placed that it is in equilibrium with the pressure of the air or vapor. If $CS = s$ be the distance of the centre of gravity of the lever from the fulcrum C, $CA = a$ the arm of the weight, and Q the weight of the lever with its valve, we then have the statical moment with which the valve is pressed down by the weight $= Ga + Qs$;* if now the pressure of the gas or vapor from below $= P$, the pressure of the atmosphere from above $= P_1$, and lastly, the arm CB of the valve $= d$, we then have the statical moment with which the valve strives to lift itself up $= (P - P_1)\, d$, and by equating the moments of both:

Fig. 391.



$$Pd - P_1 d = Ga + Qs, \text{ and } P = P_1 + \frac{Ga + Qs}{d}.$$

If r represent the radius $\frac{1}{2}$ DE of the valve, p the internal and p_1 the external tension, measured by the pressure on a square inch, we then have: $P = \pi r^2 p$ and $P_1 = \pi r^2 p_1$; hence, $p = p_1 + \dfrac{Ga + Qs}{\pi r^2 d}$.

Examples.—1. If the height of mercury of a manometer, open above, is 3,5 inches, but that of the barometer 27 inches, the corresponding expansive force is then $h = b + h_1 = 27 + 3{,}5 = 30{,}5$ inches, or $p = 0{,}491 \,.\, h = 0{,}491 \,.\, 30{,}5 = 14.97$ lbs.—2. If the height of a water-manometer is 21 inches, the expansive force corresponding to this, with the height of the barometer at 27 inches, is:

$$h = 27 + \frac{21}{13{,}6} = 28{,}54 \text{ inches} = 14{,}01 \text{ lbs. English.}$$

3. If the statical moment of an unloaded safety-valve is 10 inch lbs., the statical moment of a 10 lbs. sliding weight $15 \,.\, 10 = 150$ inch lbs., the arm of the valve measured, from the valve to the fulcrum, 4 inches, and the radius of the valve 1,5 inches, then the difference of the pressures on both surfaces of the valve is:

$$p - p_1 = \frac{150 + 10}{\pi\, (1{,}5)^2 \,.\, 4} = \frac{160}{9\,\pi} = 5{,}66 \text{ lbs.}$$

Were the pressure of the atmosphere $p_1 = 14$ lbs., the tension of the air below the valve would from this amount to $p = 19{,}66$ lbs.

§ 297. *Law of Mariotte.*—The tension of gases increases with their density; the more a certain quantity of air is compressed or condensed, the greater is its tension; and the greater its tension, the more it is allowed to expand or become rarefied, the less does its expansive force exhibit itself. The ratio in which the tension and the density, or the volume of the gases, stand to each other, is expressed by the law discovered by Mariotte, and named after him. This law assumes that *the density of one and the same quantity of air or gas is proportional to its tension*; or, as the spaces which are occupied by one and the same mass are inversely proportional to the densities,

* If the weight of the lever and valve be counterpoised by a weight attached to a cord, passing upwards and over a pulley above S, the statical moment is reduced to Ga.—AM. ED.

that the volume of one and the same mass of gas is inversely as its expansive force. Accordingly, if a certain quantity of air becomes compressed to one-half its original volume, its density is therefore doubled, its tension is also as great again as at first; and on the other hand, if a certain quantity of air be expanded to one-third of its original bulk, therefore, its density reduced one-third, its elasticity will be equal to one-third only of its original tension. If atmospheric air, for example, under the piston EF of a cylinder AC, Fig. 392, be supposed to press with 15 lbs. on every square inch, it will press on the piston with a force of 30 lbs., if this piston be pushed to E_1F_1, and the enclosed air compressed to one half its original volume, and this force will amount to $3 . 15 = 45$ lbs., if the piston come to E_2F_2, and describes two-thirds of the whole height. If the area of the piston be 1 square foot, the pressure of the atmosphere against it will amount to $= 144 . 15 = 2160$ lbs.; hence, to press down the piston one-half the height of the cylinder, it will require 2160 lbs., and to push it down two-thirds of this height $2 . 2160 = 4320$ lbs. to be exerted.

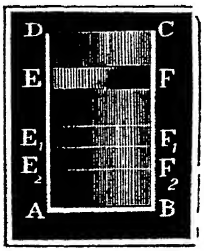


Fig. 392.

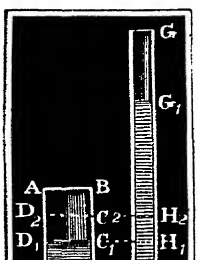


Fig. 393.

The law of Mariotte may be likewise proved by pouring mercury into the tube GH communicating with the air of a cylinder AC, Fig. 393. If a column of air AC be originally enclosed by the quantity of mercury $DEFH$, which has the same tension as the external air, and afterwards be compressed to one-half or one-fourth its volume by the addition of fresh mercury, we shall then find that the distances of the surfaces of G_1H_1, G_2H_2, &c., of mercury are equivalent to the single and treble height of the barometer b, &c., that, therefore, if we add to this the single height, corresponding to the external pressure of the air, the tension will be twice or four times as great as that due to its original volume.

The tensions are h and h_1 or p and p_1, γ and γ_1 the corresponding densities, and V and V_1 the volumes appertaining to one and the same quantity of air, we then have, according to the law laid down:

$$\frac{\gamma}{\gamma_1} = \frac{V_1}{V} = \frac{h}{h_1} = \frac{p}{p_1}; \text{ hence}$$

$$\gamma_1 = \frac{h_1}{h}\gamma = \frac{p_1}{p}\gamma \text{ and } V_1 = \frac{h}{h_1}V = \frac{p}{p_1}V.$$

From this the density and also the volume may be reduced from one tension to another.

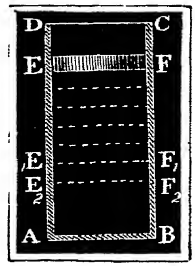
Examples.—1. If in a blowing machine, the manometer stand at 3 inches, whilst the barometer is at 28 inches, the density of the wind is $= \frac{28 + 3}{28} = \frac{31}{28} = 1{,}107$ times

that of the external air.*—2. If a cubic foot of atmospheric air, with the barometer at 28 inches, weighs $\frac{62,5}{770}$ lbs.; with the barometer at 34 inches it will weigh:

$$\frac{62.5}{770} \cdot \frac{34}{28} = \frac{21250}{21560} = 0{,}985 \text{ lbs.}$$

§ 298. The mechanical effect which must be expended to condense a certain quantity of air to a certain degree, and the effects which the air by its expansion will produce, cannot be directly assigned, because the expansive force varies at every moment of condensation or extension; we must therefore endeavor to find a special formula for the calculations of this value. Let us imagine a certain quantity of air AF, enclosed in a cylinder AC, Fig. 394, by a piston EF, and let us inquire what effect must be expended to push forward the piston through a certain space $EE_1 = FF_1$. If the original tension $= p$, and the original height of the capacity of the cylinder $= s_0$, and the tension after describing the space $EE_1 = p_1$, and the residuary volume of air $= s_1$, the proportion $p_1 : p = s_0 : s_1$ then holds true, and gives $p_1 = \frac{s_0}{s_1} p$.

Fig. 394.



While describing a very small space $E_1E_2 = x$, the tension p_1 may be regarded as invariable, and hence the mechanical effect to be expended is $= Ap_1x = \frac{Aps_0x}{s_1}$, when A represents the surface of the piston.

It follows from the properties of logarithms, that a very small magnitude $y = hyp.\ log.\ (1 + y) = 2{,}3026\ Log.\ (1 + y)$, if $hyp.\ log.$ represents the hyperbolic, and $Log.$ the common logarithms; we may consequently put

$$Aps_0 \frac{x}{s_1} = Aps_0\ hyp.\ log.\ \left(1 + \frac{x}{s_1}\right)$$
$$= 2{,}3026\ Aps_0\ log.\ \left(1 + \frac{x}{s_1}\right)$$

But now:

$$hyp.\ log.\left(1 + \frac{x}{s_1}\right) = hyp.\ log.\left(\frac{s_1 + x}{s_1}\right) = hyp.\ log.\,(s_1 + x) - hyp.\ log.\ s_1;$$

hence the elementary mechanical effect is:

$$= Aps_0\ [hyp.\ log.\ (s_1 + x) - hyp.\ log.\ s_1].$$

Let us imagine the whole space EE_1 to be made up of very small parts, such as x, and therefore put $EE_1 = nx$, we shall find the mechanical effects corresponding to all these parts, if in the last formula we substitute for

$s_1,\ s_1 + x,\ s_1 + 2\,x,\ s_1 + 3\,x, \ldots$ to $s_1 + (n-1)\,x$, and for
$s_1 + x,\ s_1 + 2\,x,\ s_1 + 3\,x$, &c., to $s_1 + nx$, or s_0,

* For comparisons with the manometer, the division of a barometer ought not to be into either inches or metres, but into 1000th parts of 1 *atmosphere*.—Am. Ed.

and by summation the whole expenditure of mechanical effect in describing the space $s_0 - s_1$:

$$L = \mathfrak{A} p s_0 \left\{ \begin{array}{l} hyp.\ log.\ (s_1 + x) - hyp.\ log.\ s_1 \\ hyp.\ log.\ (s_1 + 2x) - hyp.\ log.\ (s_1 + x) \\ hyp.\ log.\ (s_1 + 3x) - hyp.\ log.\ (s_1 + 2x) \\ \quad\vdots \qquad\qquad\qquad \vdots \\ hyp.\ log.\ (s_1 + nx) - hyp.\ log.\ [s_1 + (n-1)x] \end{array} \right.$$

$$= \mathfrak{A} p s_0 \, [hyp.\ log.\ (s_1 + nx) - hyp.\ log.\ s_1]$$

$$= \mathfrak{A} p s_0 \, (hyp.\ log.\ s_0 - hyp.\ log.\ s_1) = \mathfrak{A} p s_0 \, hyp.\ log.\ \left(\frac{s_0}{s_1}\right),$$

since one member in the one line always cancels one member in the following one.

Since $\frac{s_0}{s_1} = \frac{h_1}{h} = \frac{p_1}{p}$, this mechanical effect may be put:

$$L = \mathfrak{A} p s_0 \, hyp.\ log.\left(\frac{h}{h}\right) = \mathfrak{A} p s_0 \, hyp.\ log.\ \left(\frac{p_1}{p}\right).$$

If we put the space described by the piston $s_0 - s_1 = s$, we shall hence find that the mean force of the piston p condensing the air is in the proportion

$$\frac{h_1}{h} = \frac{p_1}{p}, \ P = \frac{L}{s} = \mathfrak{A} p \, \frac{s_0}{s} \, hyp.\ log.\ \left(\frac{p_1}{p}\right).$$

Let $\mathfrak{A} = 1$ (square foot) and $s_0 = 1$ (foot), we obtain the mechanical effect produced

$$L = p \ hyp.\ log.\ \left(\frac{p_1}{p}\right) = 2{,}3026 \ p \ log.\ \left(\frac{p_1}{p}\right).$$

This formula gives the mechanical effect which must be expended to convert a unit or cubic foot of air of a lower pressure or tension p into a higher tension p_1, and to reduce it thereby to the volume $\left(\frac{p}{p_1}\right)$ cubic feet. On the other hand:

$$L = p_1 \ hyp.\ log.\ \left(\frac{p_1}{p}\right) = 2{,}3026 \ p_1 \ log.\ \left(\frac{p_1}{p}\right)$$

expresses the effect which a unit of volume of gas gives out or produces when it passes from a higher pressure p_1 to a lower p.

To reduce by condensation a mass of air of the volume V, and tension p to the volume V_1, and the tension $p_1 = \frac{V}{V_1} p$, the mechanical effect requisite to be expended is $V p \ hyp.\ log.\ \left(\frac{V}{V_1}\right)$, and when, inversely, the volume V_1 at a tension p_1 is converted by expansion into the volume V, and into the tension $p = \frac{V_1}{V} . p_1$, the effect:

$$V p \ hyp.\ log.\ \left(\frac{V}{V_1}\right) = V_1 p_1 \ hyp.\ log.\ \left(\frac{V}{V_1}\right)$$

will be given out.

Examples.—1. If a blast converts 10 cubic feet of air per second, of 28 inches tension, into air of 30 inches tension, the effect to be expended upon this for every second will be $= 17280 \,.\, 0{,}491 \,.\, 28$ *hyp. log.* $\left(\frac{30}{28}\right) = 237565$ (*hyp. log.* 15—*hyp. log.* 14) $= 237565$ $(2{,}708050 - 2{,}639057) = 237565 \,.\, 0{,}068993 = 16390$ inch lbs. $= 1365.8$ ft. lbs.—2. If a mass of vapor in a steam engine below the surface of a piston $A = \pi 8^2 = 201$ square inches, stands 15 inches high, and with a tension of three atmospheres, pushes up the piston 25 inches, the mechanical effect evolved, and which is expended on the piston, is:

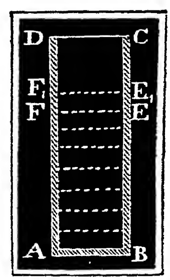
$$L = 201 \,.\, 3 \,.\, 14{,}73 \,.\, 15 \; hyp. \, log \left(\frac{15 + 25}{15}\right) = 133232 \; hyp. \, log. \, \tfrac{8}{3}$$

$= 133232 \,.\, 0{,}98083 = 130567$ inch lbs. $= 10881$ feet lbs., and the mean force of the piston, without regard to its friction and the counter pressure, is:

$$= \frac{130567}{25} = 5222 \text{ lbs.}$$

§ 299. *Strata of Air.*—Air enclosed in a vessel is at different depths of different density and tension, for the upper strata press together the lower on which they rest, so that there are only one and the same density and tension in one and the same horizontal stratum, and both increase with the depth. But in order to discover the law of this increase of density downwards, or the decrease upwards, we must adopt a method very similar to that of the former paragraph.

Fig. 395.



Let us imagine a vertical column of air AE, Fig. 395, of the transverse section $AB = 1$, and of the height $AF = s$. Let the density of the lower stratum $= \gamma$, and the tension $= p$, and the density of the upper stratum $EF = \gamma_1$, and the tension $= p_1$, we shall then have $\frac{\gamma_1}{\gamma} = \frac{p_1}{p}$. If x is the height EE_1 of the stratum E_1F, we have its weight, and hence also the diminution of its tension corresponding to:

$$y = 1 \,.\, x \,.\, \gamma_1 = \frac{x\gamma p_1}{p}, \text{ and inversely,}$$

$$x = \frac{p}{\gamma} \,.\, \frac{y}{p_1}, \text{ or as in the former paragraph:}$$

$$x = \frac{p}{\gamma} \; hyp. \, log. \left(1 + \frac{y}{p_1}\right) = \frac{p}{\gamma} \left[hyp. \, log. \, (p_1 + y) - hyp. \, log. \, p_1\right].$$

Let us put for p_1, successively

$$p_1 + y, \; p_1 + 2\,y, \; p_1 + 3\,y, \text{ \&c., to } p_1 + (n - 1)\,y,$$

and add the corresponding heights of the strata or values of x, and we shall then obtain the height of the entire column of air, as in the former §.

$$s = \frac{p}{\gamma} \left(hyp. \, log. \, p - hyp. \, log. \, p_1\right) = \frac{p}{\gamma} \; hyp. \, log. \left(\frac{p}{p_1}\right), \text{ also}$$

$$s = \frac{p}{\gamma} \; hyp. \, log. \left(\frac{b}{b_1}\right) = 2{,}302 \frac{p}{\gamma} \; log. \left(\frac{b}{b_1}\right),$$

if b and b_1 are the heights of the barometer corresponding to the tensions p and p_1.

If, inversely, the height s is given, the expansive force and density of the air corresponding to it may be calculated. It is:

$$\frac{p}{p_1} = \frac{\gamma}{_1} = e^{\frac{s\gamma}{p}}, \text{ therefore } \gamma_1 = \gamma e^{\frac{-s\gamma}{p}}, \text{ where } e = 2{,}71828$$

is the base of the hyperbolic system of logarithms.

Remark. This formula is applicable to the measurement of heights. Leaving the temperature out of consideration, we may put $s = 58604$ $. Log. \left(\frac{b}{b_1}\right)$; for the English measure $= 60000\ Log. \left(\frac{b}{b_1}\right)$.

Examples.—1. If the height of the barometer at the foot of a mountain be 28 inches, and at the top 25 inches, the height of this mountain will be:

$s = 58604\ .\ Log. \left(\frac{30}{25}\right) = 58604\ .\ 0791813 = 4640$ Prussian feet.—2. The density of the air on a mountain 10,000 feet high is: $Log.\ \frac{\gamma}{\gamma_1} = \frac{10000}{58604} = 0{,}1706$; hence, $\frac{\gamma}{\gamma_1} = 1{,}481$, and $\frac{\gamma_1}{\gamma} = \frac{1}{1{,}481} = 0{,}675$; it is therefore only $67\frac{1}{2}$ per cent. of the density of that at the foot.

§ 300. *Gay-Lussac's Law.*—Heat or temperature has an essential influence on the density and expansive force of gases. The more air enclosed in a vessel becomes heated, the greater does its expansive force exhibit itself, and the higher that the temperature of the air enclosed by a piston in a vessel is raised, the more it expands, and pushes against the piston. From the experiments of Gay-Lussac, which in later times have been repeated by Rudberg, Magnus and Regnault, it results that for equal densities the expansive force, and for equal expansive forces the volume of one and the same quantity of air increases as the temperature. We may place this law by the side of that of Mariotte, and name it, for distinction's sake, Gay-Lussac's law.

According to the latest experiments, the expansive force of a definite volume of air increases by being heated from the freezing to the boiling point, by 0,367 of its original value, or for this increase of temperature the volume of a definite quantity of air increases, the tension remaining the same, by 36,7 per cent. If the temperature is given in centigrade degrees, of which there are 100 between the freezing and boiling point, it follows that the expansion for each degree is $= 0{,}00367$, and for t degrees temperature $= 0{,}00367\ .\ t$; if we make use of Fahrenheit's thermometer, which contains between the freezing and boiling point 180°, for each degree the expansion is $= .002039$, and for t degrees temperature $= .002039\ .\ t$. This co-efficient is true only for atmospheric air; slightly greater values correspond to other gases, and even for atmospheric air, this co-efficient increases slightly with the temperature.

If a mass of air of the original volume V_0, and of the temperature (centigrade) 0°, be heated t degrees without assuming a different tension, the new volume is then $V = (1 + 0{,}00367\,t)\,V_0$, and if it acquire the temperature t_1, it will then assume the volume: $V_1 = (1 + 0{,}00367\,t_1)\,V_0$, and by dividing the ratio of the volumes:

$$\frac{V}{V_1} = \frac{1 + 0{,}00367\,t}{1 + 0{,}00367\,t_1},$$

on the other hand, the corresponding ratio of density:

$$\frac{\gamma}{\gamma_1} = \frac{V_1}{V} = \frac{1 + 0,00367\, t_1}{1 + 0,00367\, t}.$$

If, moreover, a change take place in the tensions, if p_0 is the tension at zero, p that at the temperature t, and p_1 that at t_1, we then have:

$$V = (1 + 0,00367\, t)\frac{p_0}{p}\, V_0, \text{ further } V_1 = (1 + 0,00367\, t_1)\ \frac{p_0}{p_1}\ V_0,$$

hence:

$$\frac{V}{V_1} = \frac{1 + 0,00367\, t}{1 + 0,00367\, t_1} \cdot \frac{p_1}{p}, \text{ and } \frac{\gamma}{\gamma_1} = \frac{1 + 0,00367\,.\, t_1}{1 + 0,00367\,.\, t} \cdot \frac{p}{p_1}, \text{ or,}$$

$$\frac{\gamma}{\gamma_1} = \frac{1 + 0,00367\,.\, t_1}{1 + 0,00367\,.\, t} \cdot \frac{b}{b_1}.$$

Example. If a mass of air, of 800 cubic feet, and of 10 lbs. tension, and 10° (centigrade) temperature, is raised by the blast, and by the warming apparatus of a blast-furnace to a tension of 19 lbs. and to a temperature of 200°*, it will at length assume the greater volume:

$$V_1 = \frac{1 + 0,00367\,.\,200}{1 + 0,00367\,.\,10} \cdot \frac{15}{19} \cdot 800 = \frac{1,734}{1,0367} \cdot \frac{12000}{19} = 1056 \text{ cubic feet (Prussian).}$$

§ 301. *Density of the Air.*—By aid of the formula at the end of the former paragraph, γ may now be calculated by the density corresponding to a given temperature and tension of the air. By accurate weighings and measurements we have the weight of a cubic metre of atmospheric air at a temperature of 0°, and 0,76 metre height of barometer = 1,2995 kilogrammes. Since a cubic foot (Prussian) = 0,030916 cubic metre and 1 kilogramme = 2,13809 lbs. The density of the air for the relations given is: = 0,030916 . 2,13809 . 1,2995 = 0,08590 lbs. If now the temperature is = t^0 cent., the density for the French measure: $\gamma = \dfrac{1,2995}{1 + 0,00367\, t}$ kilogrammes; and for the Prussian measure $\gamma = \dfrac{0,08590}{1 + 0,00367\,.\, t}$ lbs., and for the English: $\gamma = \dfrac{0,081241}{1 + 0,00204\, t}$ lbs. If now the expansive force varies from the mean, if, for example, the height of the barometer is not 0,76 metres, but b, we shall obtain:

$$\gamma = \frac{1,2995}{1 + 0,00367\,.\, t} \cdot \frac{b}{0,76} = \frac{1,71\,.\, b}{1 + 0,00367\, t} \text{ kilog.}$$

or if b, as is commonly the case, be given in Paris inches:

$$\gamma = \frac{0,003058\,.\, b}{1 + 0,00367\, t} \text{ lbs.}$$

Very often the expansive force is expressed by the pressure p, on a square centimetre or square inch, for this reason the factor $\dfrac{p}{1,0336}$, or $\dfrac{p}{14,73}$, or $\dfrac{p}{15,05}$ must be introduced, and it then follows that:

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* 10° C. = 50 F., and 200° C. = 392° F.,—the co-efficient will then be .002039.—Am. Ed.

$$\gamma = \frac{1,2995}{1+0,00367\,.\,t} \cdot \frac{p}{1,0336} = \frac{1,2572p}{1+0,00367\,.\,t} \text{ kilog. or}$$

$$\gamma = \frac{0,08565}{1+0,00367\,.\,t} \cdot \frac{p}{15,05} = \frac{0,005691p}{1+0,00367\,.\,t} \text{ lbs. Prussian.}$$

For the same temperature and tension, the density of steam is $\frac{5}{8}$ of the density of atmospheric air, for which reason we have for steam:

$$\gamma = \frac{0,8122}{1+0,00367\,t} \cdot \frac{p}{1,0336} = \frac{0,78577}{1+0,00367\,t} \text{ kilog. or}$$

$$\gamma = \frac{0,05353}{1+0,00367\,.\,t} \cdot \frac{p}{15,05} = \frac{0,003557p}{1+0,00367\,.\,t} \text{ lbs. Prussian.}$$

$$= \frac{0,050775}{1+0,00204\,t} \cdot \frac{p}{14,73} = \frac{0,003447p}{1+0,00204\,t} \text{ lbs. English.}$$

Examples.—1. What weight has the wind contained in a cylindrical regulator of 40 feet length and 6 feet width, at a temperature of 10° and 18 lbs. pressure? The density of this wind is:

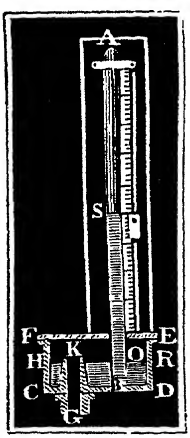
$$= \frac{0,005691\,.\,18}{1,0367} = \frac{0,10244}{1,0367} = 0,0988 \text{ lbs. (Prussian)};$$

the capacity of the regulating vessel is $= \pi\,.\,3^2\,.\,40 = 1131$ cubic feet; hence, the quantity of wind $= 0,0988\,.\,1131 = 112$ lbs.—2. A steam engine uses per minute 500 cubic feet of vapor, of 107° C. temperature and 36 inches pressure, how many pounds of water are required for the generation of this steam? The density of this steam is:

$$= \frac{0,05353}{1+0,00367.107^{\circ}} \cdot \frac{36}{28} = \frac{0,05353\,.\,36}{1,393\,.\,28} = 0,0494 \text{ lbs.};$$

hence, the weight of 500 cubic feet, or the weight of the corresponding quantity of water, $= 500\,.\,0,0494 = 24,7$ lbs.

Fig. 396.



§ 302. By aid of the results obtained in the last paragraph, the theory of the air manometer may be explained. This instrument consists of a barometer tube of uniform bore AB, Fig. 396, filled above with air and below with mercury, and of a vessel CE likewise containing mercury, which is put in communication with the gas or vapor whose tension we wish to find. From the height of the columns of mercury and of air, the expansive force may be estimated as follows. The instrument is commonly so arranged, that the mercury in the tube stands at the same level as the mercury in the cistern, when the temperature of the enclosed air $t = 50°$ (10° C.), and the tension in the space EH equal to the mean atmospheric pressure $b = 0,76$ metres $= 30$ inches.

But if for a height of the barometer b, from EH a column of mercury h_1 has ascended into the tube, and the length of the column of the residuary air is h_2, we have then its tension

$$= \frac{h_1 + h_2}{h_2}\, b, \text{ and hence } b_1 = h_1 + \frac{h_1 + h_2}{h_2}\, b.$$

If a change of temperature takes place, the temperature from observation of h_1 and h_2 is not as at first $= t$, but t_1, we then have the tension of the column of air:

$$AS = \frac{h_1 + h_2}{h_2} \cdot \frac{1+0,00367 \cdot t_1}{1+0,00367 \cdot t} \cdot b,$$

and hence the height of the barometer in question:

$$b_1 = h_1 + \frac{h_1 + h_2}{h_2} \cdot \frac{1+0,00367 \cdot t_1}{1+0,00367 \cdot t} \cdot b.$$

For $b = 28$ inches (Paris), and $t = 10°$ C., it follows that

$$b_1 = h_1 + 27\,(1 + 0,00367\; t_1)\,\frac{h}{h_2},\text{ whereby } h = h_1 + h_2,$$

represents the whole length of the tube measured to the surface of mercury HR.

From the height of the barometer b_1, it follows that the pressure on the square inch (Prussian) is

$$p = \frac{15,6}{28}\,h_1 + 15,6 \cdot \frac{27}{28}\,(1 + 0,00367\; t_1 \cdot \frac{h}{h_2} = 0,538\; h_1$$

$$+ 14,51\,(1+0,00367\; t)\,\frac{h}{h_2}\text{ lbs.}$$

Example.—If an air manometer, of 25 inches length, at 21° temperature C., indicates a column of air of 12 inches in length, then the corresponding height of the barometer is:
$b_1 = 25 - 12 + 27\,(1 + 0,00367 \,.\, 21)\,\frac{25}{12} = 13 + 9 \,.\, 1,07707 \,.\, \frac{25}{4}$
$= 13 + 60,58 = 73,58$ inches, and the pressure on a square inch
$= 0,538 \,.\, 73,58 = 39,59$ lbs.

SECTION VI.

DYNAMICS OF FLUID BODIES.

CHAPTER I.

THE GENERAL LAWS OF THE EFFLUX OF WATER FROM VESSELS.

§ 303. *Efflux.*—The doctrine of the efflux of fluids from vessels constitutes the first principal division of hydro-dynamics. We distinguish first between the efflux of *air* and the efflux of *water*, and then again between the efflux under *uniform* and that under *variable pressure*. We next treat of the efflux of water under constant pressure. The pressure of water may be assumed as constant when the same quantity of water is admitted on one side as flows out from the other, or when the quantity of water flowing out in a certain time is small compared with the size of the vessel. The chief problem, whose solution will be here treated, is that of determining the *discharge* through a given *orifice*, under a given pressure, in a definite time.

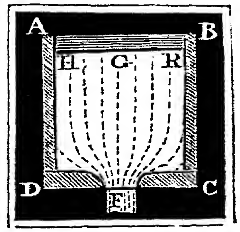
If the discharge in each second $= Q$, we then have the expenditure, after the lapse of t seconds, under variable pressure: $Q_1 = Qt$. But to obtain the efflux per second, it is necessary to know the dimensions of the orifice, and the velocity of the particles of fluid issuing from it. For the sake of simplicity of investigation, we shall for the present assume that the particles of water flow out in straight and parallel lines, and on this account form a *prismatic vein* or stream of fluid. If, now, F be the transverse section of the vein and v the velocity of the water, or of each particle of water, the discharge per second will form a prism of the base F and height v, and, therefore, Q will be $= Fv$ units of volume and $G = Fv\gamma$ units of weight, γ being the density of the water or the effluent liquid.

Examples.—1. If water flows through a sluice, of 1,7 square feet aperture, with a 14 feet velocity, the discharge will be $Q = 14 \,.\, 1,7 = 23,8$ cubic feet, and hence the discharge per hour will be $= 23,8 \,.\, 3600'' = 85680$ cubic feet.—2. If 264 cubic feet of water were to be discharged through an orifice of 5 square inches in 3 minutes and 10 seconds, the mean velocity of efflux would amount to

$$v = \frac{Q_1}{Ft} = \frac{264}{\frac{5}{144} \,.\, 190''} = \frac{264 \,.\, 144}{5 \,.\, 190} = 40 \text{ feet.}$$

§ 304. *Velocity of Efflux.*—Let us suppose a vessel AC, Fig. 397, filled with water, having a horizontal exit orifice, F, rounded from the inside, which forms but a small part of the transverse section or bottom surface CD. Let the *head of water* FG, supposed invariable during the efflux, $= h$, the velocity of efflux $= v$, and the discharge in each second $= Q$, and therefore its weight $= Q\gamma$. The mechanical effect which this mass of water produces by sinking from the height h, is $= Q h \gamma$, and the mechanical effect which the effluent mass $Q\gamma$ accumulates in its transit from a state of rest into that of the velocity v, is $\frac{v^2}{2g} Q\gamma$ (§ 71). If no loss of mechanical effect take place in its passage through the orifice, both mechanical effects will be equal, and therefore $h\ Q\gamma = \frac{v^2}{2g}\ Q\gamma$; *i. e.*, $h = \frac{v^2}{2g}$, and, inversely, $v = \sqrt{2\,g\,h}$, or in feet, $h = 0{,}0155\ v^2$, and $v = 8{,}02\ \sqrt{h}$.

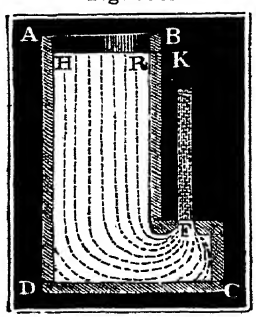
Fig. 397.



The velocity, therefore, of water issuing through an orifice is equivalent to the final velocity of a body falling freely from the height of the water.

The correctness of this law may be proved by the following experiments. If we apply an orifice directed upwards to the vessel AC, Fig. 398, the jet FK will ascend vertically, and nearly attain the level HR of the water in the vessel, and we may assume that it would exactly attain this height, were all resistances, such as those of the air, friction at the sides of the vessel, disturbances from the descending water, &c., entirely removed. But since a body ascending to a perpendicular height h, has the initial velocity $v = \sqrt{2\,g\,h}$ (§ 17), it accordingly follows that the velocity of efflux is $v = \sqrt{2\,g\,h}$.

Fig. 398.



For a different head of water h_1, the velocity is $v_1 = \sqrt{2\,g\,h_1}$, hence we have $v : v_1 = \sqrt{h} : \sqrt{h_1}$; *therefore, the velocities of efflux are to each other as the square roots of the heads of water.*

Examples.—1. The discharge which takes place in each second through an orifice 10 inches square, under a pressure of 5 feet, is:

$$Q = Fv = 10 \,.\, 12 \sqrt{2\,g\,h} = 120 \,.\, 8{,}02 \sqrt{5} = 962{,}4 \,.\, 2{,}236 = 2151{,}9 \text{ cubic inches.}$$

2. That 252 cubic inches may be discharged through an orifice of 6 square inches in each second, the head of water required is:

$$h = \frac{1}{2g}\left(\frac{Q}{F}\right)^2 = 0{,}0155 \left(\frac{252}{6}\right)^2 = 0{,}0155\ (42)^2 = 27{,}34 \text{ inches.}$$

§ 305. *Velocity of Influx and Efflux.*—If water flows in with a certain velocity c, the mechanical effect $\frac{c^2}{2g}$, corresponding to the ve-

locity due to the height $h_1 = \frac{c^2}{2g}$, must be added to the mechanical effect $h \cdot Q\gamma$; hence we have to put:

$$(h + h_1)\, Q\gamma = \frac{v^2}{2g}\, Q\gamma, \text{ or } h + h_1 = \frac{v^2}{2g},$$

and therefore the velocity of efflux:

$$v = \sqrt{2\,g\,(h + h_1)} = \sqrt{2\,g\,h + c^2}.$$

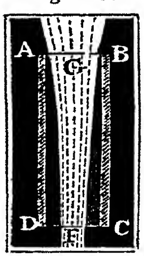
Since the quantity of water flowing into a vessel kept constantly full is as great as that Q which flows out, we may put $Gc = Fv$, where G represents the area of the transverse section HR (Fig. 397) of the water pouring in. Accordingly if we put $c = \frac{F}{G}\, v$, we shall then obtain:

$$h = \frac{v^2}{2g} - \left(\frac{F}{G}\right)^2 \frac{v^2}{2g} = \left[1 - \left(\frac{F}{G}\right)^2\right] \frac{v^2}{2g};$$

and hence: $$v = \frac{\sqrt{2\,g\,h}}{\sqrt{1 - \left(\frac{F}{G}\right)^2}}.$$

According to this formula, the velocity increases, the greater the ratio of the sections $\frac{F}{G}$ becomes; the velocity is least, viz., $= \sqrt{2gh}$, if the transverse section F of the orifice of efflux is small compared with the transverse section G of the orifice of influx, and it approximates more and more to an infinitely great velocity, the smaller the difference is between these orifices. If $F = G$, therefore $\frac{F}{G} = 1$, then $v = \frac{\sqrt{2\,g\,h}}{0} = \infty$, and therefore also $c = \infty$. This infinite value must be understood to express that the water must flow to and from a bottomless vessel AC, Fig. 399, with an infinite velocity, in order that the stream of fluid CF may entirely fill up the orifice of discharge F. If we put $v = \frac{Gc}{F}$, we shall then obtain:

Fig. 399.



$$h = \left[\left(\frac{G}{F}\right)^2 - 1\right] \frac{c^2}{2g}, \text{ hence } F = \frac{G}{\sqrt{1 + \frac{2\,g\,h}{c^2}}},$$

which expression indicates that the transverse section F of the stream flowing out, for an infinite velocity of influx, is constantly less than the transverse section G of the stream flowing in, and hence, that the discharging orifice is not quite filled when it it is greater than

$$\frac{G}{\sqrt{1 + \frac{2\,g\,h}{c^2}}}.$$

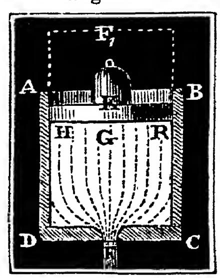
Remark. The accuracy of this formula, given by Daniel Bernoulli,
$v = \frac{\sqrt{2gh}}{\sqrt{1-\left(\frac{F}{G}\right)^2}}$, has of late been brought into doubt by many philosophers. I have endeavored to prove how unfounded are the objections made, in an article "Efflux," in the "Allgemeinen Maschinenencyclopädie."

Example. If water runs from a circular orifice, 5 inches in width, in the bottom of a prismatic vessel of 60 square inches transverse section, under a pressure of 6 feet, the velocity is then:

$$v = \frac{8{,}02\sqrt{h}}{\sqrt{1-\left(\frac{25\,\pi}{4\,.\,60}\right)^2}} = \frac{8{,}02\,.\,2{,}449}{\sqrt{1-(0{,}327)^2}} = \frac{19{,}641}{\sqrt{0{,}8931}} = \frac{19{,}641}{0{,}945} = 20{,}78 \text{ feet.}$$

§ 306. *Velocity of Efflux, Pressure, and Density.*—The above formulæ are only true if the pressure of the air on the fluid surface is as great as its pressure against the orifice; but if these pressures are different from one another, we have then to extend these formulæ. If the upper surface *HR*, Fig. 400, is pressed by a piston *K* with a force P_1, which case, for example, presents itself in that of the fire-engine, we may then imagine it to be replaced by the pressure of a column of water. If h_1 be the height of this column, and γ the density of the liquid, we may therefore put $P_1 = G h_1 \gamma$. If we substitute for h the head of water augmented by $h_1 = \frac{P_1}{G\gamma}$, $h + h_1 = h + \frac{P_1}{G\gamma}$, we then obtain for the velocity of efflux:

Fig. 400.



$v = \sqrt{2\,g\left(h + \frac{P_1}{G\gamma}\right)}$, when, moreover, we suppose $\frac{F}{G}$ to be very small. If, further, we represent the pressure on each unit of surface of G by p_1, we have more simply $\frac{P_1}{G} = p_1$, and hence

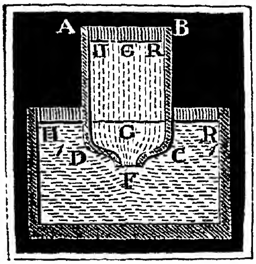
$v = \sqrt{2\,g\left(h + \frac{p_1}{\gamma}\right)}$. Again, if we represent the pressure of water at the level of the orifice by p, we may then put

$p = \left(h + \frac{p_1}{\gamma}\right)\gamma$; therefore, $h + \frac{p_1}{\gamma} = \frac{p}{\gamma}$, whence $v = \sqrt{2\,g\,\frac{p}{\gamma}}$.

The velocity of efflux, therefore, increases as the square root of the pressure on each unit of surface, and inversely as the square root of the density of the fluid. Under equal pressures, therefore, a fluid of a density represented by 4, runs out half as fast as one of density 1. Since the air is 770* times lighter than water, it would,

* According to the experiments of Prout, 100 cubic inches of air at 60° temperature and 30 inches barometer, weigh 31,0117 grains troy, and consequently at 32° a cubic foot will weigh $31{,}0117 \times \frac{521}{493} \times 17{,}28 = 566{,}1$ grains; and since a cubic foot of water weighs 62,5 lbs. avoirdupois $= 62{,}5 \times 7000 = 437500$ grains, the relative weights of

Fig. 401.



if it were an inelastic body, flow out $\sqrt{770} = 27\frac{3}{4}$ faster than water. If the water does not flow freely, but under water, in consequence of a counter-pressure, a diminution of the velocity of efflux then takes place. If the mouth of the vessel AC, Fig. 401, is the depth $FG = h$ below the surface of the upper water HR, and $FG_1 = h_1$ below the surface H_1R_1 of the lower water, we then have the pressure downwards $p = h\gamma$, and the counter pressure upwards $p_1 = h_1\gamma$, hence the force of efflux is: $p - p_1 = (h - h_1)\gamma$, and the velocity of efflux

$$v = \sqrt{2g\left(\frac{p - p_1}{\gamma}\right)} = \sqrt{2g(h - h_1)}.$$

For efflux under water the difference of level $h - h_1$ between the surfaces must be regarded as the head of water.

If the water on the side of the outer orifice be pressed by the force p, and on the side of the inner orifice or of the surface of water by the force p_1, we have then generally:

$$v = \sqrt{2g\left(h + \frac{p_1 - p}{\gamma}\right)}.$$

Examples.—1. If the piston of a 12 inch cylinder, or that of a fire-engine, were pressed down with a force of 3000 lbs., and there were no obstacles in the tubes or pipes, the water would then issue through the mouth-piece of the tube and be directed vertically upwards with a velocity:

$$v = \sqrt{2g\frac{p_1}{\gamma}} = \sqrt{2g\frac{P_1}{G\gamma}} = 8{,}02\sqrt{\frac{3000}{\frac{\pi}{4}\,.\,62{,}5}} = 8{,}02\sqrt{\frac{600\,.\,4}{\pi\;12{,}5}}$$

$= 62{,}697$ feet, and ascend to the height $h = 0{,}0155\,.\,v^2 = 60{,}93$ feet.—2. If water rushes into a rarefied space; for example, into the condenser of a steam-engine, whilst it is pressed from above or on its exposed surface by the atmosphere, the last formula $v = \sqrt{2g\left(h + \frac{p_1 - p}{\gamma}\right)}$ for the velocity of efflux is then to be applied. If the head of water $h = 3$ feet, and the external barometer stand at 27 inches, and the internal at 4 Paris inches, we shall now have $\frac{p_1 - p}{\gamma} = 27 - 4 = 23$ Paris inches $= \frac{23}{12}\;1{,}035 = 1{,}9837$ Prussian feet $= 2{,}042$ English feet, or a column of water $= 13{,}5\,.\,2{,}042 = 27{,}57$ feet, and the velocity of the water rushing into the vacuum $v = 8{,}02\,.\,\sqrt{3 + 27{,}57} = 44.34$ feet.—3. If the water in the feed-pipe of a steam-engine boiler stands 12 feet above the surface of the water in the boiler, and the pressure of steam be 20 lbs., and the pressure of the atmosphere only 15 lbs. on the square inch, the velocity with which the water enters into the boiler will be:

$$v = 8{,}02\sqrt{12 + \frac{(15-20)\,.\,144}{62{,}5}} = 8{,}02\sqrt{12 - \frac{5\,.\,144}{62{,}5}} = 8{,}02\sqrt{0{,}48} = 5{,}55 \text{ feet.}$$

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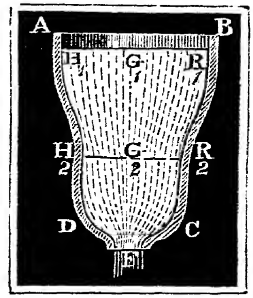
air and water at that temperature is $\frac{566{,}1}{437500} = \frac{1}{772}$. At 60° the relation will be $535{,}88 : 437500 = 1 : 816$. And as $\sqrt{816} = 28{,}5$, the velocity of efflux will, under this condition, be $\frac{0.75}{27.75}$ part more rapid than in that supposed in the text.—Am. Ed.

§ 307. *Hydraulic Pressure.*—When water enclosed in a vessel is in motion, it then presses more feebly against the sides than when at rest. We must, therefore, distinguish the hydrodynamic or hydraulic pressure from the hydrostatic pressure of water. If p_1 be the pressure on each unit of surface $H_1R_1 = G_1$, Fig. 402, p the pressure without the orifice F, and h the head of water FG_1, we then have for the velocity of efflux

Fig. 402.



$v = \sqrt{2g\left(h + \frac{p_1 - p}{\gamma}\right)} \div \sqrt{1 - \left(\frac{F}{G_1}\right)^2}$, or

$h + \frac{p_1 - p}{\gamma} = \left[1 - \left(\frac{F}{G_1}\right)^2\right]\frac{v^2}{2g}$; if, further, in another transverse section $H_2R_2 = G_2$, which lies at a height $FG_2 = h_1$ above the orifice, the pressure $= p_2$, we then have likewise:

$$h_1 + \frac{p_2 - p}{\gamma} = \left[1 - \left(\frac{F}{G_2}\right)^2\right]\frac{v^2}{2g}.$$

If we subtract one expression from the other, it then follows that:

$$h - h_1 + \frac{p_1 - p_2}{\gamma} = \left[\left(\frac{F}{G_2}\right)^2 - \left(\frac{F}{G_1}\right)^2\right]\frac{v^2}{2g},$$

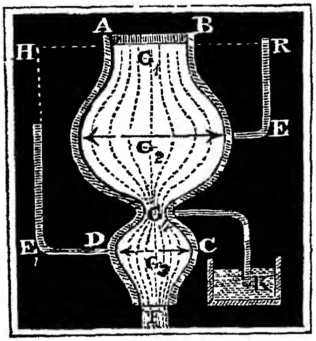
or, if the head of water G_1G_2 of the stratum $H_2R_2 = G_2$ be represented by h_2, the measure of the hydraulic pressure of water at H_2R_2 is:

$$\frac{p_2}{\gamma} = h_2 + \frac{p_1}{\gamma} - \left[\left(\frac{F}{G_2}\right)^2 - \left(\frac{F}{G_1}\right)^2\right]\frac{v^2}{2g}.$$

But now $\frac{Fv}{G_1}$ is the velocity v_1 of the water at the surface G_1, and $\frac{Fv}{G_2}$ the velocity v_2 of the water at the section G_2; hence more simply we may put $\frac{p_2}{\gamma} = \frac{p_1}{\gamma} + h_2 - \left(\frac{v_2^2}{2g} - \frac{v_1^2}{2g}\right)$.

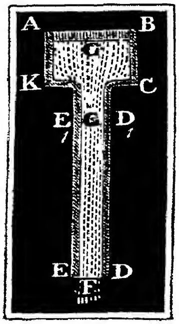
Therefore, from this it follows that the hydraulic head of water $\frac{p_2}{\gamma}$ *at any place in the vessel is equivalent to the hydrostatic head of water* $\frac{p_1}{\gamma} + h_2$ *diminished by the difference of the height due to the velocity at this point, and at the place of entrance.* If the upper surface of the water G_1 is great, we may neglect the velocity of influx, and hence may put $\frac{p_2}{\gamma} = \frac{p_1}{\gamma} + h_2 - \frac{v_2^2}{2g}$, and the *hydraulic head of water is less by the height due to the velocity than the hydrostatic head of water.* The faster, therefore, water flows in conduit pipes, the less it presses against the sides of the pipes. From this cause, pipes very often burst, or begin to leak, when its motion in them is checked, or when the pipes are stopped up, &c.

Fig. 403.



By means of the apparatus of efflux $ABCD$, Fig. 403, we may have ocular demonstration of the difference between hydraulic and hydrostatic pressure. If we carry upwards a small tube ER from the transverse section G_2, it will become filled with water, which will ascend in it above the level of the fluid surface if G_2 is $>$ G_1, therefore $v_2 < v_1$; for as the pressure p_1 on the fluid surface is counteracted by the pressure of the air against the mouth of the tube, we may put for the height which measures the pressure at G_2 viz. $x = \frac{p_2}{\gamma} = h_2 - \left(\frac{v_2^2}{2g} - \frac{v_1^2}{2g}\right)$, and therefore x is $> h_2$ if $\frac{v_2^2}{2g}$ is $< \frac{v_1^2}{2g}$. If, on the other hand, the transverse section G_3 be $< G_1$, and the water therefore flow through G_3 quicker than through G_1, we shall then have the height of the column of water in the small tube E_1 whose inner orifice is at G_3, $y = h_3 - \left(\frac{v_3^2}{2g} - \frac{v_1^2}{2g}\right)$ less than h_3, and hence it will not reach to the level HR of G_1. Again, if G_4 be very small, and therefore the corresponding velocity v_4 very great, then $\frac{v_4^2}{2g} - \frac{v_1^2}{2g}$ may be $> h_4$, and hence the corresponding hydraulic head of water z may be negative, *i. e.* the air may press more from without than the water from within. A column of water will therefore ascend in the tube E_2K, which is inserted below, and whose outer orifice is under water, which in conjunction with the pressure of the water, will balance that of the external atmosphere. If this small tube be short, the water, which may be colored for this purpose, will ascend from the vessel K underneath, through the tube, enter the reservoir of efflux, and will arrive at F and be discharged.

Fig. 404.

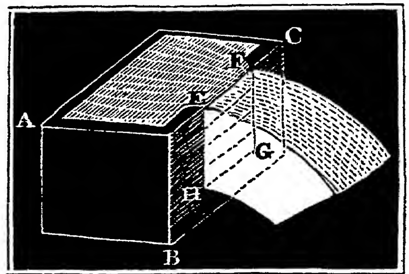


Remark. If the discharging vessel ACE, Fig. 404, consists of a wide reservoir AC and of a narrow vertical tube CE, the hydraulic pressure at all places in this tube is then negative. If we disregard the pressure of the atmosphere p_1, the pressure of the water in the vicinity of the mouth F may be put $= 0$, because the whole head of water here $GF = h$ will be expended in generating the velocity $v = \sqrt{2gh}$; on the other hand, at a place D_1E_1 at the height $G_1G = h_1$ below the surface of water, the hydraulic pressure $= h_1 - h = -(h-h_1)$ negative; if, therefore, a hole be bored in this tube, no water will run out, but air will rather be drawn in, which will arrive at F and flow out. This negative pressure will be greatest directly below the water, in G because h_2 is there least.

§ 308. By means of the formula $Q = Fv = F\sqrt{2gh}$, the discharge issuing in one second can only then be calculated directly when the orifice is horizontal, because here only the velocity throughout the whole

transverse section F is the same; but if the transverse section of the orifice has an inclination to the horizon, for example, if it is at the side of the vessel, the particles of water at different depths will then flow out with different velocities, and the discharge Q can no longer be considered as a prism, and hence, therefore, the formula $Q = Fv = F\sqrt{2gh}$ cannot be applied directly. The most simple case of this kind is presented in the efflux through a cut in the side of a vessel, or in what is called a weir, Fig. 405. This cut forms a rectangular aperture of efflux $EFGH$, whose breadth $EF = GH$ is represented by b, and height $EH = FG$ by h. If we divide this surface bh by horizontal lines into a great number n of equally broad laminæ, we may suppose the velocity in each of these to be the same. Since the heads of water of these laminæ from above downwards are $\frac{h}{n}, \frac{2h}{n}, \frac{3h}{n}$, &c., we then have the corresponding velocities

Fig. 405.



$\sqrt{2g\frac{h}{n}}, \sqrt{2g\cdot\frac{2h}{n}}, \sqrt{2g\cdot\frac{3h}{n}}$; and since, further, the area of a lamina $= b\cdot\frac{h}{n} = \frac{bh}{n}$, we then have the discharges:

$\frac{bh}{n}\sqrt{2g\frac{h}{n}}, \frac{bh}{n}\sqrt{2g\cdot\frac{2h}{n}}, \frac{bh}{n}\sqrt{2g\cdot\frac{3h}{n}}$, &c.; consequently the discharge through the entire section:

$$Q = \frac{bh}{n}\left(\sqrt{2g\frac{h}{n}} + \sqrt{2g\cdot\frac{2h}{n}} + \sqrt{2g\cdot\frac{3h}{n}} + \ldots\right)$$

$$= \frac{bh\sqrt{2gh}}{n\sqrt{n}}(\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n}).$$

But now:

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n}, \text{ or}$$

$$1^{\frac{1}{2}} + 2^{\frac{1}{2}} + 3^{\frac{1}{2}} + \ldots + n^{\frac{1}{2}} = \frac{n^{1+\frac{1}{2}}}{1+\frac{1}{2}} = \tfrac{2}{3}n^{\frac{3}{2}} = \tfrac{2}{3}n\sqrt{n};$$

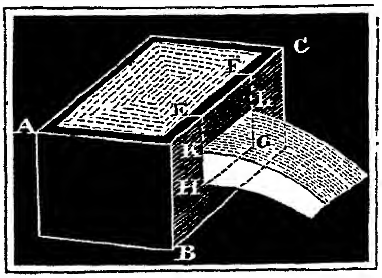
hence, the discharge required is:

$$Q = \frac{bh\sqrt{2gh}}{n\sqrt{n}}\cdot\tfrac{2}{3}n\sqrt{n} = \tfrac{2}{3}bh\sqrt{2gh} = \tfrac{2}{3}b\sqrt{2gh^3}.$$

If by the term *mean velocity* (v) *be understood that velocity which must subsist at all places, that as much water, in consequence, does issue as with the variable velocities of efflux within the whole profile;* we may then put: $Q = bh\cdot v$, and, consequently, $v = \frac{2}{3}\sqrt{2gh}$, i. e.

the mean velocity of water issuing through a rectangular cut in the side of a vessel is $\frac{2}{3}$ of the velocity at the sill or lower edge of the cut.

Fig. 406.



If the rectangular aperture of efflux KG, Fig. 406, with horizontal sill does not reach the surface of the water, we may find the discharge by regarding the aperture as the difference of the two cuts $EFGH$ and $EFLK$. Hence, if h_1 is the depth HE of the lower, and $KE = h_2$ that of the upper edge, we then have the discharge from these apertures $\frac{2}{3}\, b \sqrt{2\, g\, h_1^3}$ and $\frac{2}{3}\, b \sqrt{2 g h_2^3}$, and hence the quantity of water for the rectangular orifice $GHKL$:

$Q = \frac{2}{3}\, b \sqrt{2\, g\, h_1^3} - \frac{2}{3}\, b \sqrt{2\, g\, h_2^3} = \frac{2}{3} b \sqrt{2g}\,(h_1^{\frac{3}{2}} - h_2^{\frac{3}{2}})$, and the mean velocity of efflux:

$$v = \frac{Q}{b(h_1 - h_2)} = \frac{2}{3} \sqrt{2\, g} \,.\, \frac{h_1^{\frac{3}{2}} - h_2^{\frac{3}{2}}}{h_1 - h_2}.$$

If h is the mean head of water $\frac{h_1 + h_2}{2}$, or the depth of the centre of the orifice below the surface of water, and a the height of the orifice $HK = h_1 - h_2$, we may then put:

$$v = \frac{2}{3} \sqrt{2g} \,.\, \frac{\left(h + \frac{a}{2}\right)^{\frac{3}{2}} - \left(h - \frac{a}{2}\right)^{\frac{3}{2}}}{a}, \text{ or approximately:}$$

$$= \left[1 - \tfrac{1}{96}\left(\frac{a}{h}\right)^2\right] \sqrt{2gh.}$$

Example. If a rectangular orifice is 3 feet wide and $1\frac{1}{4}$ feet high, and the lower edge lies $2\frac{3}{4}$ feet below the surface of water, the discharge is then:

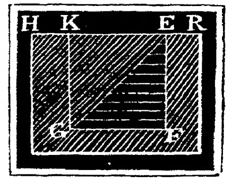
$Q = \frac{2}{3} . 8{,}02 . 3\, (2{,}75^{\frac{3}{2}} - 1{,}5^{\frac{3}{2}}) = 16{,}04\, (4{,}560 - 1{,}837) = 16{,}04 \,.\, 2{,}723 = 43{,}67$ cubic feet. From the formulæ of approximation the mean velocity of efflux is:

$v = \left[1 - \frac{1}{96}\left(\frac{1{,}25}{2{,}125}\right)^2\right] . \; 8{,}02 \sqrt{2{,}125} = (1 - 0{,}0036) \,.\, 11{,}685 = 11{,}685 - 0{,}042 =$ 11,643 feet, and hence the discharge $Q = 3 \,.\, \frac{5}{4} \,.\, 11{,}643 = 43{,}65$ cubic feet.

Remark.—If the cut in the side is inclined to the horizon at an angle δ, we shall then have to substitute the height of the aperture $\frac{h_1 - h_2}{sin.\, \delta}$ for its vertical projection, whence we must put $Q = \frac{2}{3}\, \frac{b \sqrt{2g}}{sin.\, \delta} (\sqrt{h_1^3} - \sqrt{h_2^3})$. If the transverse section of the reservoir parallel to the aperture be not considerably greater than the section of the aperture, we shall then have to take into account the velocity $v_1 = \frac{F}{G}\, v$ with which the water flows to it, and for this reason put:

$$Q = \frac{2}{3}\, b \sqrt{2g} \left[\left(h_1 + \frac{v_1^2}{2g}\right)^{\frac{3}{2}} - \left(h_2 + \frac{v_1^2}{2g}\right)^{\frac{3}{2}}\right].$$

Fig. 407.



§ 309. *Triangular Lateral Orifice.* — Besides rectangular lateral orifices, we have in practice triangular and circular. Let us first consider the efflux through a triangular orifice EFG, Fig. 407, with horizontal base, whose vertex E lies in the surface of the water. Let the base $FG = b$, and the height $EF = h$, let us divide the last into n equal parts, and carry through the points of division lines parallel to the base, we then resolve the entire surface into small elements of the areas:

$$\frac{b}{n} \cdot \frac{h}{n}, \frac{2b}{n} \cdot \frac{h}{n}, \frac{3b}{n} \cdot \frac{h}{n}, \&c.,$$

and the heads of water:

$$\frac{h}{n}, \frac{2h}{n}, \frac{3h}{n}, \&c.$$

The discharges for these are:

$$\frac{bh}{n^2}\sqrt{2g\frac{h}{n}}, \frac{2bh}{n^2}\sqrt{2g \cdot \frac{2h}{n}}, \frac{3bh}{n^2}\sqrt{2g \cdot \frac{3h}{n}}, \&c.,$$

and we obtain the discharge for the whole orifice:

$$Q = \frac{bh}{n^2}\sqrt{2g\frac{h}{n}}(1 + 2\sqrt{2} + 3\sqrt{3} + \ldots + n\sqrt{n})$$

$$= \frac{bh\sqrt{2gh}}{n^2\sqrt{n}}(1 + 2^{\frac{3}{2}} + 3^{\frac{3}{2}} + \ldots + n^{\frac{3}{2}}),$$

or since the series in the parenthesis

$$= \frac{n^{\frac{3}{2}+1}}{\frac{3}{2}+1} = \tfrac{2}{5}n^{\frac{5}{2}},\ Q = \tfrac{2}{5}\,bh\sqrt{2gh} = \tfrac{2}{5}\,b\sqrt{2gh^3}.$$

If the base of the orifice EGK lies in the surface and the vertex lower by h, we then have the discharge $\frac{2}{3}\,bh\sqrt{2gh}$ flowing through the rectangle $EFGK$,

$$Q_1 = \tfrac{2}{3}\,bh\sqrt{2gh} - \tfrac{2}{5}\,bh\sqrt{2gh} = \tfrac{4}{15}\,bh\sqrt{2gh}.$$

Through the trapezium $ABCD$, Fig. 408, whose upper base $AB = b_1$, lies in the surface of the water, and whose lower base is $CD = b_2$, and height $DE = h$, we may find the discharge by regarding the orifice as composed of a rectangle and two triangles, viz.,

$$Q = \tfrac{2}{3}\,b_2h\sqrt{2gh} + \tfrac{4}{15}(b_1 - b_2)\,h\sqrt{2gh} = \tfrac{2}{15}(2\,b_1 + 3\,b_2)\,h\sqrt{2gh}.$$

Fig. 408.

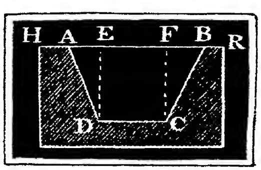
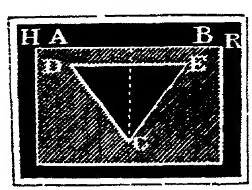


Fig. 409.



Further, the discharge for a triangle CDE, Fig. 409, of the base $DE = b_1$, and of the height h_1, and whose vertex C is distant h from

the surface of water: Q = discharge through ABC less the discharge through AE

$$= \tfrac{4}{15}\, bh \sqrt{2\,gh} - \tfrac{2}{15}\,(2\,b + 3\,b_1)\, h_1 \sqrt{2\,gh_1}$$
$$= \tfrac{2}{15} \sqrt{2\,g}\,[2\,b\,(h^{\frac{3}{2}} - h_1^{\frac{3}{2}}) - 3\,b_1 h_1^{\frac{3}{2}}].$$

As the breadth $AB = b$ may be determined by the proportion $b : b_1 = h : (h - h_1)$, it follows that

$$Q = \frac{2\sqrt{2g}\,.\,b_1}{15}\left(\frac{2\,h\,(h^{\frac{3}{2}} - h_1^{\frac{3}{2}})}{h - h_1} - 3\,h_1^{\frac{3}{2}}\right)$$
$$= \frac{2\sqrt{2\,g}b_1}{15}\left(\frac{2\,h^{\frac{5}{2}} - 5\,h\,h_1^{\frac{3}{2}} + 3\,h_1^{\frac{5}{2}}}{h - h_1}\right).$$

Lastly, for a triangle ACD, Fig. 410, whose vertex lies above the base, the quantity discharged is

$$Q = \tfrac{2}{3}\sqrt{2g}\,.\,b_1\,(h^{\frac{3}{2}} - h_1^{\frac{3}{2}}) - \frac{2\sqrt{2g}\,.\,b_1}{15}\left(\frac{2\,h^{\frac{5}{2}} - 5\,h\,h_1^{\frac{3}{2}} + 3\,h_1^{\frac{5}{2}}}{h - h_1}\right)$$
$$= \frac{2\sqrt{2g}\,.\,b_1}{15}\left(\frac{3\,h^{\frac{5}{2}} - 5\,h_1\,h^{\frac{3}{2}} + 2\,h_1^{\frac{5}{2}}}{h - h_1}\right).$$

Fig. 410.

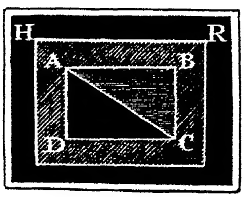
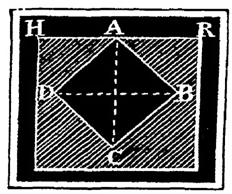


Fig. 411.



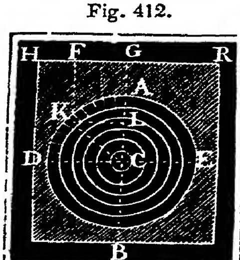
Example. What quantity of water flows through the square $ABCD$, Fig. 411, whose vertical diagonal AC = 1 foot, if the angular point A reaches the surface of the water? The upper half of this square gives the expenditure:

$Q = \frac{4}{5}\, b \sqrt{2\,gh^3} = \frac{4}{5}\, 1\,.\,8{,}02 \sqrt{\frac{1}{8}} = 0{,}4\,.\,8{,}02\,.\,0{,}353 = 3{,}21\,.\,0{,}353 = 1{,}1331$ cubic feet, but the lower water expenditure:

$$Q_1 = \frac{2\,b\sqrt{2g}}{15}\left(\frac{2\,h^{\frac{5}{2}} - 5\,h\,h_1^{\frac{3}{2}} + 3\,h_1^{\frac{5}{2}}}{h - h_1}\right) = \frac{2\,.\,8{,}02}{15}\left(\frac{2 - 5\,(\frac{1}{2})^{\frac{3}{2}} + 3\,(\frac{1}{2})^{\frac{5}{2}}}{1 - \frac{1}{2}}\right)$$

$= \frac{32{,}08}{15}\,(2 - 1{,}7678 + 0{,}5303) = \frac{32{,}08\,.\,0{,}7625}{15} = 1{,}6307$ cubic feet; the discharge through the entire orifice is $Q = 1{,}1331 + 1{,}6307 = 2{,}7638$ cubic feet.

Fig. 412.



§ 310. *Circular Lateral Orifices.*—The discharge through a circular orifice AB, Fig. 412, may be determined by an approximate formula in the following manner. Let us decompose the orifice by concentric circles into equally small annuli, and each annulus into very small elements, which may be regarded as parallelograms. If now r is the radius of such an annulus, b its breadth and n the number of its elements, we have the magnitude of an element

$K, = \frac{2\pi r b}{n}$. If h is the depth CG of the centre C below the surface of water HR, and ϕ the angle ACK, by which an element K is distant from the highest point A of the annulus, we have then the head of water of this element:

$$KF = CG - CL = h - r\cos.\phi,$$

and hence the discharge of this element

$= \frac{2\pi r b}{n}\sqrt{2g(h - r\cos.\phi)}$. Now $\sqrt{h - r\cos.\phi}$

$$= \sqrt{h}\left[1 - \tfrac{1}{2}\frac{r}{h}\cos.\phi - \tfrac{1}{8}\left(\frac{r}{h}\right)^2\cos.\phi^2 + ..\right]$$

$$= \sqrt{h}\left[1 - \tfrac{1}{2}\frac{r}{h}\cos.\phi - \tfrac{1}{16}\left(\frac{r}{h}\right)^2(1 + \cos.2\phi) + \dots\right];$$

hence the discharge of an element:

$$= \frac{2\pi r b}{n}\sqrt{2gh}\left[1 - \tfrac{1}{2}\cdot\frac{r}{h}\cos.\phi - \tfrac{1}{16}\left(\frac{r}{h}\right)^2(1 + \cos.2\phi) - \dots\right].$$

The discharge of an entire annulus is now known, if we put in the parenthesis for 1, $n \cdot 1 = n$, for $\cos.\phi$ the sum of all the cosines of ϕ from $\phi = 0$ to $\phi = 2\pi$, and for the cosine of 2ϕ, the sum of all the cosines of 2ϕ from $2\phi = 0$ to $2\phi = 4\pi$. But as the sum of all the cosines of a complete circle is $= 0$, these cosines vanish, and the discharge for the annulus:

$$= \frac{2\pi r b}{n}\sqrt{2gh}\left[n - \tfrac{1}{16}\left(\frac{r}{h}\right)^2 \cdot n - ..\right]$$

$$= 2\pi r b\sqrt{2gh}\left[1 - \tfrac{1}{16}\left(\frac{r}{h}\right)^2 - \dots\right].$$

If now for b we substitute $\frac{r}{m}$, and for r, $\frac{r}{m}, \frac{2r}{m}, \frac{3r}{m}$, to $\frac{mr}{m}$, we then obtain the discharge of all the annuli which make up the circular surface, and lastly, the quantity of efflux of the whole circle

$$Q = 2\pi r\sqrt{2gh}\left[\frac{r}{m^2}(1 + 2 + 3 + .. + m)\right.$$

$$\left. - \tfrac{1}{16}\frac{r^3}{m^4h^2}(1^3 + 2^3 + 3^3 + .. + m^3)\right]$$

$$= 2\pi r\sqrt{2gh}\cdot\left(\frac{r}{m^2}\cdot\frac{m^2}{2} - \tfrac{1}{16}\frac{r^3}{m^4h^2}\cdot\frac{m^4}{4}\right)$$

$$= \pi r^2\sqrt{2gh}\left[1 - \tfrac{1}{32}\left(\frac{r}{h}\right)^2 - ..\right],$$

or more accurately:

$$Q = \pi r^2\sqrt{2gh}\left[1 - \tfrac{1}{32}\left(\frac{r}{h}\right)^2 - \tfrac{5}{1024}\left(\frac{r}{h}\right)^4 - ..\right].$$

If the circle reaches the surface of the water, then

$$Q = \tfrac{987}{1024}\pi r^2\sqrt{2gh} = 0{,}964\, F\sqrt{2gh},$$

if F represents the area of the circle.

It is besides easy to conceive that in all cases where the head of

water at the centre is equal to or greater than the diameter, we may put the whole series $= 1$, and take $Q = F \sqrt{2gh}$. This rule may also be applied to other orifices, and, therefore, in all cases where the centre of gravity of an orifice lies at least as deep below the fluid surface as the figure is high, the depth h of this point may be regarded as the head of water, and Q put $= F \sqrt{2gh}$.

If we consider that the mean of all the cosines of the first quadrant $= \frac{\pi}{4}$, and that all the cosines of the second $= -\frac{\pi}{4}$, and likewise that the mean of the first and of the second vanishes, we may then, by the method adopted above, find the discharge of the upper semicircle:

$$Q_1 = \frac{\pi r^2}{2} \sqrt{2gh} \left[1 - \frac{\pi}{12}\left(\frac{r}{h}\right) - \frac{1}{32}\left(\frac{r}{h}\right)^2\right],$$

and that of the lower:

$$Q_2 = \frac{\pi r^2}{2} \sqrt{2gh} \left[1 + \frac{\pi}{12}\left(\frac{r}{h}\right) - \frac{1}{32}\left(\frac{r}{h}\right)^2 \cdot\cdot\right].$$

Example. What quantity of water flows hourly through a circular orifice 1 inch in diameter, above which the fluid surface stands $\frac{1}{12}$ inch high?

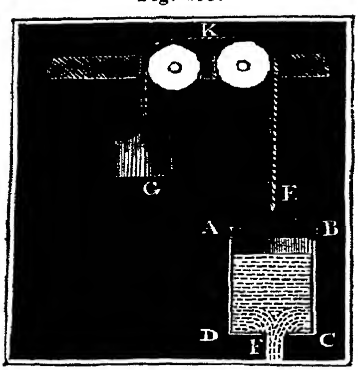
$\frac{r}{h} = \frac{6}{7}$, hence $\left(\frac{r}{h}\right)^2 = \frac{36}{49} = 0{,}735$; further, $1 - \frac{1}{32}\left(\frac{h}{r}\right)^2 = 1 - 0{,}023 = 0{,}977$, and consequently the discharge per second:

$Q = \frac{\pi \cdot 1^2}{4} 12 \cdot 8{,}02 \sqrt{\frac{7}{144}} \cdot 0{,}977 = \frac{\pi}{4} \cdot 8{,}02 \cdot 0{,}977 \sqrt{7} = 16{,}26$ cubic inches, which, per minute, $= 973$, and per hour $= 33{,}78$ cubic feet.

§ 311. *Discharging Vessels in Motion.*—The velocity of efflux varies if a vessel previously at rest or in uniform motion changes its condition of motion, because in this case every particle acts by its own weight, as well as by its inertia against the surrounding medium.

If we move the vessel AC, Fig. 413, upwards with a vertical accelerating force, whilst the water flows through the bottom by the hole F, an increase takes place, and if it be moved downwards vertically by an accelerating force, a diminution of the velocity of efflux ensues. If p is the accelerating force, each element of water M presses not only by its own weight Mg, but also by its inertia Mp; consequently the force of each element in the one case, must be put $(g + p) M$, and in the other $(g - p) M$, therefore instead of g, $g \pm p$. From this it follows then that $\frac{v^2}{2} = (g \pm p) h$, and hence for the velocity:

Fig. 413.



$$v = \sqrt{2(g \pm p) h}.$$

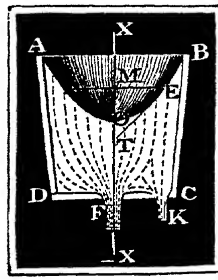
If the vessel ascends with the accelerating force g, then is

$v = \sqrt{2 \cdot 2\, gh} = 2 \sqrt{gh}$, therefore the velocity of efflux 1,414 times that of a vessel at rest. If the vessel falls by its own weight, therefore, with the accelerated motion g, there is $v = \sqrt{0} = 0$, no water therefore flows out. If the vessel moves uniformly up or down, there remains $v = \sqrt{2\, gh}$, but if it ascends with a retarded motion, then will $v = \sqrt{2\,(g - p)\, h}$, and if it descends with the same retardation, then $v = \sqrt{2\,(g + p)\, h}$.

If the vessel moves horizontally, or at an acute angle to the horizon (§ 274), the fluid surface will be inclined to the horizon, and a change in the velocity of flow will take place.

By the rotation of a vessel AC, Fig. 414, about its vertical axis $X\overline{X}$, the concave surface forms a parabolic funnel AOB, hence there will be over the middle F of the bottom a lesser head of water than at the edges, and hence the water will flow through the orifice F, in the axis, more slowly than through any other orifice K at the bottom. If h represent the head of water in the middle, then the velocity of efflux at the middle will be $= \sqrt{2\, gh}$, if y be the distance $FK = ME$ of any other orifice K from the axis, and ω the angular velocity, we shall then have the corresponding elevation of the water above the middle:

Fig. 414.



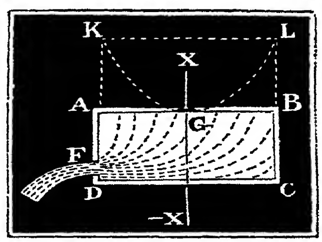
$$OM = \tfrac{1}{2}\, TM = \tfrac{1}{2}\, ME \text{ cotang. } T = \tfrac{1}{2}\, y \cdot \frac{\omega^2 y}{g} = \frac{\omega^2 y^2}{2g} = \frac{w^2}{2g},$$

if w be the velocity of rotation of the orifice K. Hence then the velocity of efflux for this is

$$v = \sqrt{2g\left(h + \frac{w^2}{2g}\right)} = \sqrt{2\, gh + w^2}.$$

This formula is true for every arbitrarily shaped vessel, and also for one closed above, as AC, Fig. 415, so that the funnel cannot be formed. Its application to wheels of reaction and to turbines will be found in the sequel.

Fig. 415.



Examples.—1. If a vessel full of water AC, Fig. 413, weighs 350 lbs., and by means of a rope passing over a roller K is drawn by a weight G of 450 lbs., it will ascend with an accelerating force $p = \frac{450 - 350}{450 + 350} \cdot g = \frac{100}{800}\, g = \frac{1}{8}\, g$, and hence the velocity of efflux will be $v = \sqrt{2\,(g + p)\, h} = \sqrt{2 \cdot \frac{9}{8} \cdot gh} = \sqrt{\frac{9}{4}\, gh}$. Were the head of water $h = 4$ feet, the velocity of efflux would be $v = 1 \sqrt{9 \cdot g} = 3 \sqrt{32{,}2} = 16{,}01$ feet.—2. If a vessel AC, Fig. 415, full of water revolves so that it makes 100 revolutions per minute, if the depth of the orifice F below the surface of water in the middle amounts to 2 feet, and the distance from the axis $X\overline{X}$, 3 feet, then the velocity of efflux is

$$v = \sqrt{2\, g\, h + w^2} = \sqrt{64{,}4 \cdot 2 + \left(\frac{3 \cdot \pi \cdot 100}{30}\right)^2} = \sqrt{128{,}8 + 100 \cdot \pi^2}$$

$= \sqrt{128 + 987} = 33{,}4$ feet. If the vessel be at rest the velocity will be $v = \sqrt{128.8}$ $= 11.34$ feet.

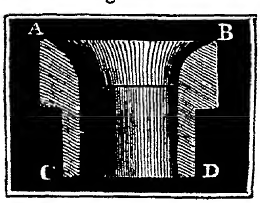
CHAPTER II.

ON THE CONTRACTION OF THE FLUID VEIN BY THE EFFLUX OF WATER THROUGH ORIFICES IN A THIN PLATE.

§ 312. *Co-efficient of Velocity.*—The laws of efflux developed in the preceding chapter accord almost entirely with experiment, so long as the head of water is not small compared with the width of the orifice, and as long as the orifice gradually widens inwards without forming corners or edges, and is close at the bottom or sides of the vessel. The experiments made by Michelotti, by Eytelwein, and by the author on this subject, with smoothly polished metallic mouth-pieces, have shown that the effective discharge, or that which actually flows out, amounts to from 96 to 98 per cent. of the theoretical quantity.

The mouth-piece *AD*, Fig. 416, represented in half its natural size, gave for a head of water of 10 feet, 97,5 per cent., for 5 ft. 96,9 per cent., and for 1 ft. 95,8 per cent. Since for this efflux the fluid vein has the same transverse section as the orifice, we must then assume that this diminution of discharge is accompanied with a loss of velocity, which is caused by the friction or adhesion of the water to the inner circumference of the orifice, and by the viscidity of the water. In what follows, we shall call the ratio of the effective velocity of efflux to that of the theoretical $= v \sqrt{2\,gh}$, the *co-efficient of velocity*, and represent it by ϕ. From this, therefore, the effective velocity of efflux in the most simple case is $v_1 = \phi\, v = \phi \sqrt{2\,gh}$, and the discharge:

Fig. 416.



$$Q = Fv_1 = \phi\, Fv = \phi\, F \sqrt{2\,gh}.$$

If we substitute for ϕ the mean value 0,97, we then obtain for the quantity in feet

$$Q = 0{,}97 \,.\, F \sqrt{2\,gh} = 0{,}97 \,.\, 8{,}02\, F \sqrt{h} = 7{,}779\, F \sqrt{h}.$$

A *vis viva* $\frac{Q\gamma}{g} \,.\, v_1^2$, is inherent in a discharge Q issuing with the velocity v_1, by virtue of which it is capable of producing the mechanical effect $Q\gamma \,.\, \frac{v_1^2}{2g}$. But since by its descent from the height $h = \frac{v^2}{2g}$, the weight $Q\gamma$ produces the mechanical effect $Q\gamma \,.\, h = Q\gamma \frac{v^2}{2g}$, it follows that by the efflux of the water, this suffers a loss

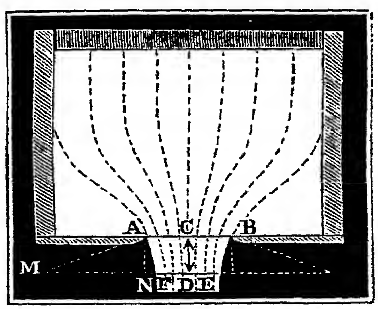
$$Q\gamma\left(\frac{v^2}{2g}-\frac{v_1^2}{2g}\right) = Q\gamma \cdot \frac{v^2}{2g}(1-\phi^2) = (1-0{,}97^2)\, Q\gamma \cdot \frac{v^2}{2g}$$
$$= 0{,}059\ Q\gamma \cdot \frac{v^2}{2g}, \text{ or } 5{,}9 \text{ per cent.}$$

Therefore, the effluent water produces by its *vis viva* 5,9 per cent. less mechanical effect, than does its weight by falling from the height h.

§ 313. *Co-efficient of Contraction.*—If water flows through an *orifice in a thin plate*, a considerable diminution of the discharge under otherwise similar circumstances takes place, whilst the particles of fluid rushing through the orifice move in convergent directions, and in this way give rise to a *contraction of the fluid vein.* The measurements of the vein made by many, and especially of late by the author, have shown that the vein at a distance which is about equal to one-half of the width of the orifice, has the greatest contraction, and a thickness equal to 0,8 that of the diameter of the orifice. If F_1 is the transverse section of the contracted vein, as also F the transverse section of the orifice, we then have from this $F_1 = (0{,}8)^2\, F = 0{,}64\, F$. The ratio $\frac{F_1}{F}$ of these transverse sections is called the *co-efficient of contraction*, and is represented by α, and accordingly, the mean value for the efflux of water through orifices in a thin plate may be put: $\alpha = 0{,}64$.

As long as we possess no more accurate knowledge on the contraction of the fluid vein, we may assume that the stream flowing through a circular orifice AB, Fig. 417, forms a body of rotation $ABEF$, whose envelope is generated by the revolution of a circular arc AF about the axis CD of the stream. Let the diameter AB of the orifice $= d$, and the distance CD of the contracted section EF, $= \frac{1}{2}\, d$, we then obtain the radius:

Fig. 417.



$MA = MF = r$ of the generating arc AF by means of the equation:

$$\overline{AN}^2 = FN\,(2\, MF - FN), \text{ or}$$
$$\frac{d^2}{4} = \frac{d}{10}\left(2r - \frac{d}{10}\right),\ r = 1{,}3\, d.$$

Orifices made after this figure of the contracted vein give pretty nearly the velocity of discharge $v_1 = 0{,}97 \cdot v$.

The contraction of the fluid vein is caused by the water which lies directly above the orifice flowing out together with that which comes to it from the sides. There takes place, therefore, in the interior of the vessel a convergence of the filaments of water, similar to that represented in the figure, and the contraction of the fluid vein con-

sists in a mere propagation of this convergence. We may convince ourselves of this motion of the water in the vicinity of the orifice by means of a glass apparatus of efflux; if we drop into the fluid minute substances which are either heavier or lighter than water, for example, such as oak saw-dust, bits of sealing wax, &c., and allow them to pass out with it from the orifice.

§ 314. *Contraction of the Fluid Vein.*—If water flows through triangular or quadrilateral orifices, and in a thin plate, the stream then assumes particular figures. The inversion of the jet, or the altered position of its transverse section with respect to that of the orifice, is very striking to the eye, in consequence of which a corner of this section comes to coincide with the middle of one side of the orifice.

Hence, from a triangular orifice ABC, Fig. 418, the section of the stream at a certain distance from the orifice forms a treble star-like vein DEF, from a quadrilateral orifice $ABCD$, Fig. 419, a star of

Fig. 418.

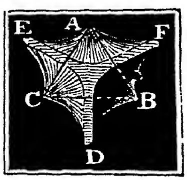


Fig. 419.

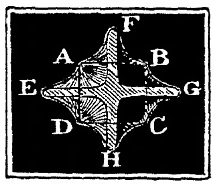
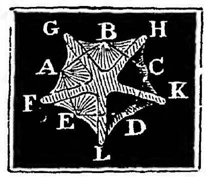
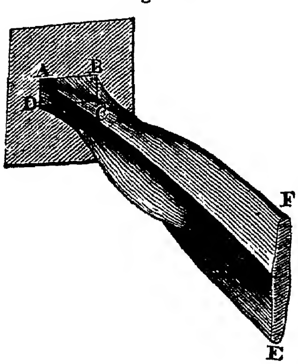


Fig. 420.



four veins $EFGH$, from a five-sided orifice $ABCDE$, Fig. 420, a star $FGHKL$, consisting of five veins. These sections vary at different distances from the orifice: at a certain distance they diminish, and at a successive one again increase; hence the vein consists of plates or ribs of variable breadth, and thereby forms, when the efflux is observed under great pressure, bulges and nodes, similar to what is seen in the cactus. If the orifice $ABCD$, Fig. 421, is rectangular; at a lesser distance from the orifice, the section will then form a cross or star; and at a greater one, it will again assume the form of a rectangle EF.

Fig. 421.



Observations on various kinds of orifices have been made by Bidone, and accurate measurements of the vein from square apertures also by Poncelet and Lesbros.* The last measurements have led to a small co-efficient of contraction 0,563. The measurements of water issuing through lesser orifices, give us, however, greater co-efficients of contraction; they show, moreover, that these are greater for elon-

* *See* Allgem. Maschinen-encyclopädie, article Ausfluss.

gated rectangles than for rectangles which approximate more to the square.

§ 315. *Co-efficient of Efflux.*—If in the flow of water through orifices in thin plates, the effective velocity were equal to the theoretical $v = \sqrt{2gh}$, we should have the effective discharge:

$$Q = \alpha F . v = \alpha F \sqrt{2gh},$$

because αF represents the transverse section of the vein at the place of greatest contraction, where the particles of water move in parallel directions. But this is by no means the case: it is shown rather by experience that Q is smaller than $\alpha F \sqrt{2gh}$, that we must therefore multiply the theoretical discharge $F \sqrt{2gh}$ by a co-efficient which is less than the co-efficient of contraction, in order to obtain the effective discharge. We must hence assume that for efflux from an orifice in a thin plate, a certain loss of velocity takes place, and therefore introduce a co-efficient of velocity ϕ, and hence put the effective velocity of efflux $v_1 = \phi v = \phi \sqrt{2gh}$. From this then we have the effective discharge: $Q_1 = F_1 . v_1 = \alpha F . \phi v = \alpha \phi F v = \alpha \phi F \sqrt{2gh}$. Again, if we call the ratio of the effective discharge to the theoretical or hypothetical quantity, the *co-efficient of efflux*, and represent it in what follows by μ, we then have:

$$Q_1 = \mu Q = \mu F v = \mu F \sqrt{2gh},$$

hence $\mu = \alpha \phi$, i. e. *the co-efficient of efflux is the product of the co-efficients of contraction and of velocity.*

Multiplied observations, but chiefly the measurements of the author, have led to this, that the co-efficient of efflux for orifices in thin plates is not constant; that for small orifices and for small velocities, it is greater than for large orifices and for great velocities: and that it is considerably greater for elongated and small orifices than for orifices which have a regular form, or which approximate to the circle.

For square orifices of from 1 to 9 square inches area, with from 7 to 21 feet head of water, according to the experiments of Bossut and Michelotti, the mean co-efficient of efflux is $\mu = 0{,}610$; for circular ones of from $\frac{1}{2}$ to 6 inches diameter, with from 4 to 21 feet head of water, $\mu = 0{,}615$, or about $\frac{8}{13}$. The single values observed by Bossut and Michelotti vary considerably from one another, but we cannot discover in them any dependence between the dimensions of the orifice and the magnitude of the head of water. From the author's experiments at a pressure of 24 inches, the co-efficient for an orifice of

0,3937 inches or	1 centimetre	diameter is	μ	= 0,628
0,7874 "	2 centimetres	"		= 0,621
1,1811 "	3 "	"		= 0,614
1,5748 "	4 "	"		= 0,607.

On the other hand, at a pressure of 10 inches for the round orifice of

1 centimetre	diameter	μ	= 0,637
2 centimetres	"		= 0,629
3 "	"		= 0,622
4 "	"		= 0,614

From these it is manifest that the co-efficient of efflux increases when the dimensions of the orifice and the head of water decrease.

If for μ we take the mean value $= 0{,}615$, and for $\alpha = 0{,}64$, we obtain the co-efficient of velocity for the efflux through orifices in a thin plate, $\varphi = \frac{\mu}{\alpha} = 0{,}96$, therefore, nearly as great as for efflux through rounded or conoidal orifices.

Remark 1. Buff's experiments (see Poggendorf's Ann. Band 46), show that the co-efficient of efflux for small orifices and for small heads of water or velocities is considerably greater than for large or mean orifices and velocities. An orifice of 2,084 lines diameter, gave for $1\frac{1}{2}$ inch pressure, $\mu = 0{,}692$, for 35 inches $\mu = 0{,}644$; on the other hand, an orifice of 4,848 lines for $4\frac{1}{2}$ inches pressure $\mu = 0{,}682$, and for 29 inches $\mu = 0{,}653$.

Remark 2. According to the author's experiments, the co-efficients for efflux under water are about $1\frac{1}{3}$ per cent. less than for efflux in air.

§ 316. *Rectangular Lateral Orifices.*—The most accurate experiments on efflux through large rectangular lateral apertures are those made at Metz by Poncelet and Lesbros. The widths of these orifices were two decimetres, (nearly 8 inches); the depths, however, varied from one centimetre to two decimetres. In order to produce perfect contraction, a brass plate of four millimetres, $= 0{,}1575$ inches, thickness was used for these orifices. From the results of their experiments, these experimenters have calculated by interpolation the tables at the end of this paragraph for the co-efficients which may be used for the measurement or calculation of the discharge.

If b be the breadth of the orifice, and if h_1 and h_2 are the heads of water above the lowest, and above the uppermost horizontal edge of the orifice, we then have, from § 308, the discharge: $Q = \frac{2}{3}\, b \sqrt{2g}\, (h_1^{\frac{3}{2}} - h_2^{\frac{3}{2}})$. But if we substitute the height of the aperture a, and the mean head of water $h = \frac{h_1 + h_2}{2}$, we then have approximately $Q = \left(1 - \frac{a^2}{96\, h^2}\right) ab \sqrt{2gh}$, and hence the effective discharge $Q_1 = \mu\, Q = \left(1 - \frac{a^2}{96\, h^2}\right) \mu\, ab \sqrt{2gh}$. If, further, we put $\left(1 - \frac{a^2}{96\, h^2}\right) \mu = \mu_1$, we have then simply $Q_1 = \mu_1 ab \sqrt{2gh}$, and in order to allow of our calculating by this simple or general formula of efflux, not only the values of μ, but also those of μ_1 are given in the following tables.

Since the water in the vicinity of the orifice is in motion, it stands lower directly before the aperture than at a greater distance from the plate in which the aperture is made; on this account two tables have been compiled, the one for heads of water measured at a greater distance from the orifice, and the other for those measured immediately at the side in which the orifice lies. It may be seen, moreover, from both tables, although with certain variations, that the co-efficients of efflux increase the lower the orifice is, and the less the head of water.

If the orifices have different breadths, we are compelled, so long as we have no further experiments, still to use the co-efficients of these tables in like manner for the calculation of the discharge. If, further, the orifices are not rectangular, we must determine their mean breadth and mean depth, and introduce into the calculation the co-efficients corresponding to these dimensions. Lastly, it is always preferable to measure the head of water at a certain distance from the side in which the orifice lies, and to use the first table, than directly at the orifice where the surface of water is curved and less tranquil, than a little above it.

Examples. 1.—What quantity of water flows through a rectangular aperture, 2 decimetres broad and 1 decimetre deep, if the surface of water is $1\frac{1}{2}$ metre above the upper edge? Here $b = 0,2$; $a = 0,1$, $h = \frac{h_1 + h_2}{2} = \frac{1,6 + 1,5}{2} = 1,55$; hence the theoretical discharge $Q = 0,1 \,.\, 0,2 \sqrt{2g} \,.\, \sqrt{1,55} = 0,02 \,.\, 4,429 \,.\, 1,245 = 0,1103$ cubic metre. But now Table I. gives for $a = 0,1$ and $h_2 = 1,5$, $\mu_1 = 0,611$, hence the effective discharge $Q_1 = 0,611 \,.\, 0,1103 = 0,0674$ cubic metre—2. What discharge corresponds to a rectangular orifice in a thin plate of 8 inches breadth, 2 inches depth, with a 15 inches head of water above the upper edge?* The theoretical discharge is $Q = \frac{2}{3} \,.\, \frac{1}{6} \,.\, 7,906 \sqrt{\frac{4}{3}} = 0,8784 \,.\, 1,1547 = 1,014$ cubic feet. But now 2 inches is about 0,05 metre, and 15 inches about 0,4 metre; hence, according to the table $a = 0,05$ and $h_2 = 0,4$, the corresponding co-efficient $\mu_1 = 0,628$ is to be taken, and the quantity of water sought is $Q_1 = 0,628 \,.\, 1,014 = 0,637$ cubic feet.—3. If the breadth $= 0,25$, the depth $= 0,15$, and the head of water $h_2 = 0,045$ metre, then is $Q = 0,25 \,.\, 0,15 \,.\, 4,429 \,.\, \sqrt{0,12} = 0,166 \,.\, 0,3464 = 0,0575$ cubic metre. To the height 0,15 corresponds for $h_2 = 0,04$, the mean value: $\mu_1 = \frac{0,582 + 0,603}{2} = 0,5925$, and $h_2 = 0,05$, $\mu_1 = \frac{0,585 + 0,605}{2} = 0,595$; but since h_2 is given $= 0,045$, we must then substitute the new mean $\frac{0,5925 + 0,595}{2} = 0,594$ for the co-efficient of efflux, and we therefore obtain the discharge sought: $Q_1 = 0,594 \,.\, 0,0575 = 0,03415$ cubic metre.

* In using the following tables, the English measures will be furnished with the proper co-efficients by employing the first, or left-hand column, in which to find the height h, and the column under the number of inches answering to the height of orifice a.—Am. Ed.

TABLE I.

The co-efficients for the efflux through rectangular orifices in a thin vertical plate, from Poncelet and Lesbros. The heads of water are measured at a certain distance back from the orifice, or at a point where the water may be considered as still.

Head of water, or distance of the surface of water from the upper side of the orifice.		HEIGHT OF ORIFICE.					
Eng. in.	Metres.	0,20^m or 8 in.	0,10^m or 4 in.	0,05^m or 2 in.	0,03^m or 1,18 in.	0,02 or ,8 in.	0,01^m or ,4 in.
0,00	0,000	"	"	"	"	"	"
0,19	0,005	"	"	"	"	"	0,705
0,39	0,010	"	"	0,607	0,630	0,660	0,701
0,57	0,015	"	0,593	0,612	0,632	0,660	0,697
0.78	0,020	0,572	0,596	0,615	0,634	0,659	0,694
1,18	0,030	0,578	0,600	0,620	0,638	0,659	0,688
1,57	0,040	0,582	0,603	0.623	0,640	0,658	0,683
1,97	0,050	0,585	0,605	0,625	0,640	0,658	0,679
2,36	0,060	0,587	0,607	0,627	0,640	0,657	0,676
2,75	0,070	0,588	0,609	0,628	0,639	0,656	0,673
3,14	0,080	0,589	0,610	0,629	0,638	0,656	0,670
3,54	0,090	0,591	0,610	0,629	0,637	0,655	0,668
3,93	0,100	0,592	0,611	0,630	0,637	0,654	0,666
4,72	0,120	0,593	0,612	0,630	0,636	0,653	0,663
5,51	0,140	0,595	0,613	0,630	0,635	0,651	0,660
6,29	0,160	0,596	0,614	0,631	0,634	0,650	0,658
7,08	0,180	0,597	0,615	0,630	0,634	0,649	0,657
7,87	0,200	0,598	0,615	0,630	0,633	0,648	0,655
9,84	0,250	0,599	0,616	0,630	0,632	0,646	0,653
11,81	0,300	0,600	0,616	0,629	0,632	0,644	0,650
15,75	0,400	0,602	0,617	0,628	0,631	0,642	0,647
19,68	0,500	0,603	0,617	0,628	0,630	0,640	0,644
23,62	0,600	0,604	0,617	0,627	0,630	0,638	0,642
27,56	0,700	0,604	0,616	0,627	0,629	0,637	0,640
31,49	0.800	0,605	0,616	0,627	0,629	0,636	0,637
35,43	0,900	0,605	0,615	0,626	0,628	0,634	0,635
39,37	1,000	0,605	0,615	0,626	0,628	0,633	0,632
43,30	1,100	0,604	0,614	0,625	0,627	0,631	0,629
47,24	1,200	0,604	0,614	0,624	0,626	0,628	0,626
51,18	1,300	0,603	0,613	0,622	0,624	0,625	0,622
55,11	1,400	0,603	0,612	0,621	0,622	0,622	0,618
59,05	1,500	0,602	0,611	0,620	0,620	0,619	0,615
62,99	1,600	0,602	0,611	0,618	0,618	0,617	0,613
66,93	1,700	0,602	0,610	0,617	0,616	0,615	0,612
70,86	1,800	0,601	0,609	0,615	0,615	0,614	0,612
74,80	1,900	0,601	0,608	0,614	0,613	0,612	0,611
78,74	2,000	0,601	0,607	0,613	0.612	0,612	0,611
118,11	3,000	0,601	0,603	0,606	0,608	0,610	0,609

TABLE II.

Co-efficients of efflux through rectangular orifices in a vertical plate, from Poncelet and Lesbros. The heads of water are measured directly at the orifice.

Head of water, or distance of the surface of water from the upper side of the orifice.		HEIGHT OF ORIFICE.					
Eng. in.	Metres.	$0{,}20^{m}$ or 8 in.	$0{,}10^{m}$ or 4 in.	$0{,}05^{m}$ or 2 in.	$0{,}03^{m}$ or 1,18 in.	$0{,}02^{m}$ or ,8 in.	$0{,}01^{m}$ or ,4 in.
0,00	0,000	0,619	0,667	0,713	0,766	0,783	0,795
0,19	0,005	0,597	0,630	0,668	0,725	0,750	0,778
0,39	0,010	0,595	0,618	0,642	0,687	0,720	0,762
0,57	0,015	0,594	0,615	0,639	0,674	0,707	0,745
0,78	0,020	0,594	0,614	0,638	0,678	0,697	0,729
1,18	0,030	0,593	0,613	0,637	0,659	0,685	0,708
1,57	0,040	0,593	0,612	0,636	0,654	0,678	0,695
1,97	0,050	0,593	0,612	0,636	0,651	0,672	0,686
2,36	0,060	0,594	0,613	0,635	0,647	0,668	0,681
2,75	0,070	0,594	0,613	0,635	0,645	0,665	0,677
3,14	0,080	0,594	0,613	0,635	0,643	0,662	0,675
3,54	0,090	0,595	0,614	0,634	0,641	0,659	0,672
3,93	0,100	0,595	0,614	0,634	0,640	0,657	0,669
4,72	0,120	0,596	0,614	0,633	0,637	0,655	0,665
5,51	0,140	0,597	0,614	0,632	0,636	0,653	0,661
6,29	0,160	0,597	0,615	0,631	0,635	0,651	0,659
7,08	0,180	0,598	0,615	0,631	0,634	0,650	0,657
7,87	0,200	0,599	0,615	0,630	0,633	0,649	0,656
9,84	0,250	0,600	0,616	0,630	0,632	0,646	0,653
11,81	0,300	0,601	0,616	0,629	0,632	0,644	0,651
15,75	0,400	0,602	0,617	0,629	0,631	0,642	0,647
19,68	0,500	0,603	0,617	0,628	0,630	0,640	0,645
23,62	0,600	0,604	0,617	0,627	0,630	0,638	0,643
27,56	0,700	0,604	0,616	0,627	0,629	0,637	0,640
31,49	0,800	0,605	0,616	0,627	0,629	0,636	0,637
35,43	0,900	0,605	0,615	0,626	0,628	0,634	0,635
39,37	1,000	0,605	0,615	0,626	0,628	0,633	0,632
43,30	1,100	0,604	0,614	0,625	0,627	0,631	0,629
47,24	1,200	0,604	0,614	0,624	0,626	0,628	0,626
51,18	1,300	0,603	0,613	0,622	0,624	0,625	0,622
55,11	1,400	0,603	0,612	0,621	0,622	0,622	0,618
59,05	1,500	0,602	0,611	0,620	0,620	0,619	0,615
62,99	1,600	0,602	0,611	0,618	0,618	0,617	0,613
66,93	1,700	0,602	0,610	0,617	0,616	0,615	0,612
70,86	1,800	0,601	0,609	0,615	0,615	0,614	0,612
74,80	1,900	0,601	0,608	0,614	0,613	0,613	0,611
78,74	2,000	0,601	0,607	0,614	0,612	0,612	0,611
118,11	3,000	0,601	0,603	0,606	0,608	0,610	0,609

§ 317. *Wiers.*—If water flows through *wiers*, or through notches in a thin plate, as, for example, *FB*, Fig. 422, the fluid vein then suffers a contraction on three sides, by which a diminution of the discharge is effected, since the quantity discharged from these orifices is $Q_1 = \frac{2}{3} \mu\, b\, h \sqrt{2\, gh}$. But here the head of water $EH = h$, or the head of water above the sill, of the wier must not be measured immediately at the sill, but at least two feet before the plate in which the orifice lies, because the fluid surface before the opening suffers a depression, which becomes greater and greater the nearer it is to the orifice, and in the plane of the orifice amounts to a quantity GR of from 0,1 to 0,25 the head of water FR, so that the thickness FG of the stream in this plane is only 0,9 to 0,75 of the head of water. Experiments instituted by many philosophers on the flow of water through notches in thin plates, have afforded a multiplicity of results, but not always of the desired accordance. The following short table contains the results of the experiments of Poncelet and Lesbros on wiers of two decimetres, or about 8 inches breadth.

Fig. 422.

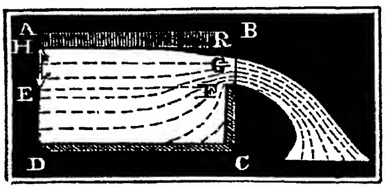


TABLE OF THE CO-EFFICIENTS OF EFFLUX FOR WIERS OF 2 DECIMETRES, = 7.87 INCHES BREADTH, ACCORDING TO PONCELET AND LESBROS.

Head of water h.	metrs. 0,01 or ,4 in	metrs. 0,02 ,8 in.	metrs 0,03 1,2 in.	metrs. 0,04 1,6 in.	metrs. 0,06 2,4 in.	metrs. 0,08 3,2 in.	metrs. 0,10 4 in.	metrs. 0,15 6 in.	metrs. 0,20 8 in.	metrs. 0,22 9 in.
Co-efficient of efflux $\mu_1 = \frac{2}{3}\mu$.	0,424	0,417	0,412	0,407	0,401	0,397	0,395	0,393	0,390	0,385

From the average of determinations, we may here put $\mu_1 = 0,4$. Experiments on wiers of greater breadth gave Eytelwein the mean $\mu_1 = \frac{2}{3}\mu = 0,42$, and Bidone $\mu_1 = \frac{2}{3} \,.\, 0,62 = 0,41$, &c. The most extensive experiments are those of d'Aubuisson and Castel. From these, d'Aubuisson asserts that for wiers whose breadth is no more than the third part of the breadth of the canal or side in which the wier lies, the mean of μ is $= 0,60$, therefore, we may put $\frac{2}{3}\mu = 0,40$: but, on the other hand, for wiers which extend over the whole side, or have the same breadth as the water-course: $\mu = 0,665$, therefore, $\mu_1 = 0,444$; lastly, for other relations between the breadth of the wier and that of the canal, the co-efficients of efflux are very different, and lie between 0,58 and 0,66. Experiments made by the author, reduce the variability of these co-efficients to certain laws (§ 322).

[During his investigations in the summer of 1845, to determine the relative value of the several sources for supplying water to the city of Boston, the editor had an opportunity of making extensive series of

experiments on the passage of water through wiers of 1, 2, and 3,01 feet in breadth, and from 0,066 foot to 2,087 feet in depth above the bottom of the notch. The water was measured in a cubical box, 6 feet on a side, to which was attached, on the exterior, a glass gauge tube with a scale extending to the top of the receptacle. In like manner, a gauge tube was inserted in the dam which contained the notch, and several feet distant from it, with the 0 of its scale accurately adjusted to the level of the bottom of the notch. A scale sliding vertically was placed immediately over the centre of the wier by which the depth over the edge of the notch-board could be ascertained. The reservoir from which the water was drawn was at least 6 times as wide as the opening of the notch. The following are some of the co-efficients ($\mu_1 = \frac{2}{3}\mu$) for the several breadths of wier:

1.—Wier 3,01 feet in breadth, cut in 2 inch planks—

Full depth h, over bottom of notch.	Co-efficients of discharge $= \frac{2}{3}\mu$.	Depression of surface at the notch.
0,075 feet	0,3667	0,021 feet
0,189 "	0,3794	0,040 "
0,280 "	0,3973	0,070 "
0,316 "	0,4211	0,079 "
0,360 "	0,4307	0,086 "
0,480 "	0,4349	0,097 "
0,545 "	0,4376	0,120 "
0,689 "	0,4301	0,149 "
0,755 "	0,4294	0,155 "
0,801 "	0,4208	0,158 "
1,023 "	0,4129	0,167 "

2. Wier 2 feet wide, in a 1 inch board—

0,199 feet	0,4195	unc. "
1,020 "	0,4344	0,196 "
1,062 "	0,4408	0,206 "
1,232 "	0,4477	0,228 "
1,280 "	0,4460	0,230 "

3. Wier 1 foot wide, in 1 inch board—

0,329 feet	0,4144	unc. "
0,333 "	0,4166	unc. "
0,339 "	0,4191	unc. "
0,352 "	0,4265	0,068 "
0,360 "	0,4265	0,070 "
2,060 "	0,4149	0,118 "
2,087 "	0,4130	0,125 "

The "full depth over the notch" here signifies, of course, that of the general level of the reservoir, above the edge of the wier. In each case it will be observed that the above co-efficients increase with the increase of depth up to a certain point, and then diminish gradually as far as the observations were extended. As these experiments were made with a view to determine an important practical and economical question, they were conducted with great care, and are believed

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·thy of reliance as the basis of computation for works on an scale.]

;—1. A wier, 0,25 metre broad and 0,15 head of water, gives per second the $Q = 0{,}393 \cdot bh\sqrt{2gh} = 0{,}393 \cdot 4{,}429 \cdot 0{,}25 \cdot (0{,}15)^{\frac{3}{2}} = 0{,}435 \cdot 0{,}0581 =$ bic metres.—2. What breadth must be given to a wier which, with a head f 8 inches, will allow 6 cubic feet per second to pass through? It is

$$= \frac{Q_1}{\mu_1\sqrt{2gh^3}} = \frac{6}{0{,}4 \cdot 8{,}02\sqrt{(\frac{2}{3})^3}} = \frac{6}{3{,}208 \cdot 0{,}5443} = 3{,}436 \text{ feet.}$$

g to Eytelwein, we take $\mu_1 = 0{,}42$, it follows that:

$$b = \frac{6}{3{,}368 \cdot 0{,}5443} = 3{,}27 \text{ feet.}$$

. *Maximum and Minimum of Contraction.*—In the flow of rough orifices in a plane side, the axis of the stream is per-ar to the surface of the side, and therefore the amount of the on is a mean, but if the axis of the orifice or of the fluid orms an acute angle with the portion of the side containing e, the contraction will be less; and if the angle between this l the inner surfaces of the edges of the aperture, be obtuse, raction will be still greater. The one case is represented in , and the other in Fig. 424. Without doubt this difference action is caused by the particles of water, which flow towards e from the sides, deviating less from their direction in the in the other case, when they pass through the orifice and form a fluid stream.

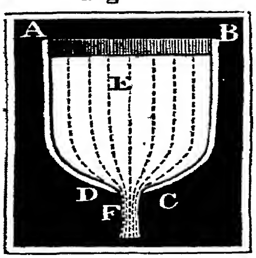


Fig. 423.

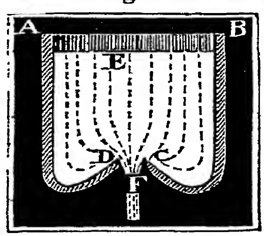


Fig. 424.

ontraction is a minimum, *i. e.* nothing, if by the gradual con- e of the side which embraces the orifice, the lateral flow is prevented, and a maximum if the side has a direction opposite

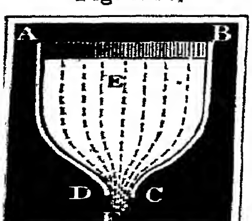


Fig. 425.

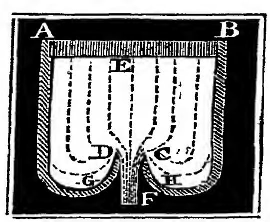


Fig. 426.

the fluid stream, so that certain particles of water must re- before arriving at the orifice. Both cases are represented

in Figs. 425 and 426. In the first case, the co-efficient of efflux is about 1, viz. 0,96 to 0,97; and in the second from the measurements of Borda, Bidone and the author, a mean of 0,53. Changes in the co-efficients of efflux through convergent sides very often present themselves in practice; they occur in dams, which are inclined to the horizon, as in Fig. 427. Poncelet found for a similar opening the co-efficient of efflux $\mu = 0{,}80$, when the board was inclined 45°, and on the other hand, $\mu = 0{,}74$ only for an inclination of $63\frac{1}{2}°$, that is, for a slope of $\frac{1}{2}$. For similar wiers, Fig. 428, where, as in the

Fig. 427.

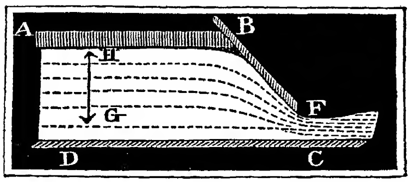
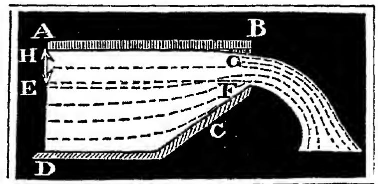


Fig. 428.



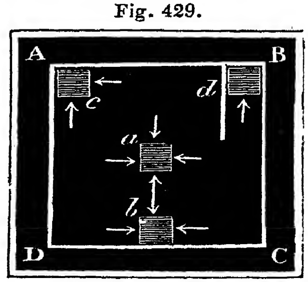
Poncelet sluice-board, contraction takes place at one side only, the author found $\mu = 0{,}70$, therefore, $\mu_1 = \frac{2}{3}\,\mu = 0{,}467$ for an inclination of 45°, and $\mu = 0{,}67$, therefore, $\mu_1 = 0{,}447$ for an inclination of $63\frac{1}{2}°$.

Example. If a sluice-board, inclined at an angle of 50°, which goes across a channel $2\frac{1}{4}$ feet broad, is drawn up $\frac{1}{2}$ foot high, and the surface of water stands 4 feet above the bottom of the channel, the height of the aperture may be put $a = \frac{1}{2}\,sin.\,50° = 0{,}3830$ feet, the head of water $h = 4 - \frac{1}{2}\,.\,0{,}3830 = 3{,}8085$ feet, and the co-efficient of efflux $\mu = 0{,}78$; hence, the discharge $Q_1 = 0{,}78\,.\,2{,}25\,.\,0{,}3830\,.\,8{,}02\,\sqrt{3{,}8085} = 10{,}49$ cubic feet (English).

§ 319. *Partial Contraction.*—We have only hitherto considered those cases where the water flows from all sides towards the aperture, and forms a contracted vein around, and we must now investigate others, where the water flows from one or more sides to the aperture, and therefore produces a stream only partially contracted. To distinguish the circumstances of contraction, we will call the case, where the vein is contracted on all sides, *general;* and the case, where it is only contracted in one part of its circumference, *partial*, or *imperfect contraction*. Partial contraction is induced when an orifice in a plane thin plate is confined by other plates in the direction of the fluid stream at one or more sides.

In Fig. 429, are represented four orifices of equal size a, b, c, d, in the bottom AC of a vessel. The contraction by efflux through the orifice a in the middle of the bottom is general, because the water can flow to it from all sides; the contraction from the efflux through b, c, d, is partial, because the water can only flow to them from one, two, or three sides. Likewise, if a rectangular lateral aperture goes to the bottom of the vessel, the contraction is then partial, because it falls away at the bottom side, if, further, the aperture of the

Fig. 429.

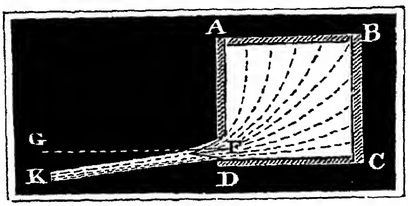


dam reaches the bottom in the lateral walls of the channel, there is then only a contraction on one side.

Partial contraction is remarkable in two respects; first, by giving an oblique direction to the stream; and, secondly, by increasing the quantity of discharge.

If the lateral aperture F, Fig. 430, reaches a second side CD, so that no contraction takes place there, the axis FK of the fluid stream becomes deflected by an angle KFG of about 9° from the normal FG to the plane of the orifice. The obliquity of the stream is much greater if two adjacent sides of the orifice have projecting borders. If the orifice has borders in two oppositely situated sides, and contraction at these prevented, such a deviation of course will not take place, but at the other side, the vein at some distance from the orifice, will spread out more than if the border were not there. Although a greater discharge is obtained by a partial contraction, we must, as a rule, endeavor to avoid this, because the fluid stream, in consequence, suffers a deviation in its direction and a greater extension in its breadth.

Fig. 430.



Experiments on the efflux of water with partial contraction have been made by Bidone and by the author. They allow us to assume that the co-efficients of efflux increase simultaneously with the ratio of the contracted part to the whole perimeter, though it is easy to perceive that this relation is different, if the perimeter is almost or entirely restricted, and the contraction almost or entirely suppressed. Let us put the ratio of this restriction to the entire perimeter $= n$, and let us represent by $\varkappa$, any number deduced from experiment, we may then, although only approximately, put the ratio of the corresponding co-efficient of efflux μ_n of partial contraction to the co-efficient of efflux of perfect contraction:

$$\frac{\mu_n}{\mu_0} = 1 + \varkappa\, n, \text{ and consequently } \mu_n = (1 + \varkappa\, n)\, \mu_0.$$

Bidone's experiments give for circular orifices $\varkappa = 0{,}128$, and for rectangular $\varkappa = 0{,}152$; the author's, however, give for the last, $\varkappa = 0{,}134$. Rectangular orifices with borders, are those which are most frequently met with in practice; we will assume for them the mean value $\varkappa = 0{,}143$, and hence put $\mu_n = (1 + 0{,}143 \,.\, n)\, \mu_0$. For a rectangular lateral orifice of the depth a and breadth b, $n = \frac{b}{2\,(a + b)}$, if the contraction on one side b is suppressed; if, for instance, this side lies in the plane of the bottom; again, $n = \frac{1}{2}$, if a side a and a side b are bordered, and $n = \frac{2\,a + b}{2\,(a + b)}$, if on one side b, and both sides a, the contraction is prevented; if, for example, the

orifice takes up the whole breadth of the reservoir, and reaches the plane of the bottom.

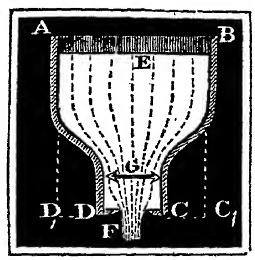
Example. What quantity of water flows through a 3 feet broad and 10 inch deep vertical aperture of a dam at a pressure of $1\frac{1}{2}$ feet above the upper side of the aperture, if the lower one coincides with the bottom of the channel, and hence there is no contraction at the bottom? The theoretical discharge is:

$$Q = \tfrac{10}{12} \cdot 3 \cdot 8{,}02 \sqrt{1{,}5 + \tfrac{5}{12}} = \tfrac{5}{2} \cdot 8{,}02 \sqrt{1{,}9166\ldots} = 28{,}11 \text{ cubic feet.}$$

According to Poncelet's table for general contraction $\mu = 0{,}604$, we have, therefore, $n = \frac{3}{2\,(3 + \frac{10}{12})} = \frac{9}{18 + 5} = \frac{9}{23}$; hence, for the above cases of partial contraction $\mu n\ (= 1 + 0{,}143 \cdot \frac{9}{23}) \cdot 0{,}604 = 1{,}056 \cdot 0{,}604 = 0{,}638$, and the effective discharge is $Q_1 = 0{,}638\, Q = 0{,}638 \cdot 28{,}11 = 18{,}14$ cubic feet.

§ 320. *Imperfect Contraction.*—The contraction of the fluid vein depends, further, upon whether the water before the orifice is tolerably still, or whether it arrives before it with a certain velocity. The quicker the water flows to the orifice, the less contracted does the vein become, and the greater is the discharge. The relations of contraction and efflux above given and investigated, have reference only to the case where the orifice lies in a large side, and it can only be assumed that the water flows to it with a small velocity; hence we must know the relations of contraction and efflux, when the transverse section of the orifice is not much less than that of the affluent water, and when, consequently, the water arrives at the orifice with a considerable velocity. In order to distinguish these two cases from one another, we shall call the contraction, where the superincumbent water is still, *perfect;* and that where it is in motion, *imperfect contraction.* The contraction, for example, is imperfect in the efflux from a vessel AC, Fig. 431, because the transverse section F of the orifice is not much smaller than that of G of the arriving water, or the area of the side CD, in which this orifice lies. If, on the other hand, the vessel had the form ABC_1D_1, and, therefore, the area of the bottom surface C_1D_1 much greater than the transverse section F of the orifice, the efflux would then go on with perfect contraction. The imperfectly contracted vein is besides distinguishable, not merely by its greater thickness from the perfectly contracted fluid vein, but also by its not having so transparent and crystalline an appearance.

Fig. 431.



If the ratio of the area of the orifice F, and the side containing the orifice G, therefore, $\frac{F}{G} = n$, the co-efficient of efflux for perfect contraction $= \mu_0$, and that for imperfect $= \mu_n$, we may with greater accuracy, according to the experiments and calculations made by the author, put:

1. For circular orifices:

$$\mu_n = \mu_0\, [1 + 0{,}04564\, (14{,}821^{\,n} - 1)], \text{ and}$$

2. For rectangular orifices:

$$\mu^n = \mu_0 \, [1 + 0{,}0760 \, (9^n - 1)].^*$$

To render the calculation easier in cases of application, the corrections $\frac{\mu_n - \mu_0}{\mu_0}$ of the co-efficient of efflux on account of imperfect contraction are compiled in the following short tables.

TABLE I.

CORRECTIONS OF THE CO-EFFICIENTS OF EFFLUX FOR CIRCULAR ORIFICES.

n	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50
$\frac{\mu_n - \mu_0}{\mu_0}$	0,007	0,014	0,023	0,034	0,045	0,059	0,075	0,092	0,112	0,134
n	0,55	0,60	0,65	0,70	0,75	0,80	0,85	0,90	0,95	1,00
$\frac{\mu_n - \mu_0}{\mu_0}$	0,161	0,189	0,223	0,260	0,303	0,351	0,408	0,471	0,546	0,613

TABLE II.

CORRECTIONS OF THE CO-EFFICIENTS OF EFFLUX FOR RECTANGULAR ORIFICES.

n	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50
$\frac{\mu_n - \mu_0}{\mu_0}$	0,009	0,019	0,030	0,042	0,056	0,071	0,088	0,107	0,128	0,152
n	0,55	0,60	0,65	0,70	0,75	0,80	0,85	0,90	0,95	1,00
$\frac{\mu_n - \mu_0}{\mu_0}$	0,178	0,208	0,241	0,278	0,319	0,365	0,416	0,473	0,537	0,608

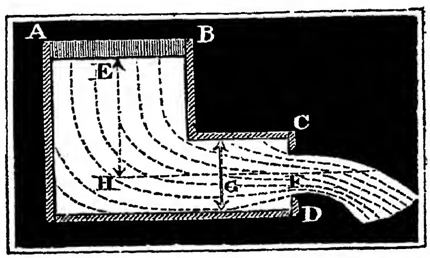
The different values of the ratio of the transverse sections $n = \frac{F}{G}$ stands above in these tables, and immediately below additions to the

* "Experiments on the Imperfect Contraction of Water," &c., Leipzic, 1843.

co-efficients of efflux, on account of imperfect contraction; for example, for the ratio of the transverse sections $n = 0,35$, *i. e.*, for the case where the area of an orifice is 35 hundredths of the area of the whole side of the orifice, we have for circular orifices $\frac{\mu_n - \mu_0}{\mu_0} = 0,075$, and for rectangular orifices $= 0,088$; therefore, the co-efficient of efflux for perfect contraction in the first case is to be made about 75 thousandths, and in the second about 88 thousandths greater to obtain the corresponding co-efficients of efflux for imperfect contraction. Were the co-efficient of efflux $\mu_0 = 0,615$, we should have in the first case $\mu_{0\ 35} = 1,075 . 0,615 = 0,661$, and in the second, $\mu_{0\ 35} = 1,088 . 0,615 = 0,669$.

Example.—What discharge does a rectangular lateral aperture F, $1\frac{1}{4}$ feet broad and $\frac{1}{2}$ foot deep, give if it be cut in a rectangular wall CD, Fig. 432, 2 feet broad and 1 foot deep, and the head of water $EH = h$ in still water amounts to 2 feet? The theoretical discharge is $Q = 1,25 . 0,5 . 8,02 \sqrt{2} = 5,012 . 1,414 = 7,086$ cubic feet, and the co efficient of efflux for perfect contraction is, according to Poncelet, $\mu_0 = 0,610$; but now the ratio of the transverse sections $n = \frac{F}{G} = \frac{1,25 . 0,5}{2 . 1} = 0,312$, and for $n = 0,312$, from Table II.,

Fig. 432.



$\frac{\mu_n - \mu_0}{\mu_0} = 0,071 + \frac{1}{5}\frac{2}{0} (0,088 - 0,071) = 0,071 + 0,004 = 0,075$; hence it follows, that the co-efficient of efflux for the present case is $\mu_{0,312} = 1,075 . \mu_0 = 1,075 . 0,610 = 0,6557$, and the discharge $Q_1 = 0,6557 . Q = 0,6557 . 6,987 = 4,581$ cubic feet.

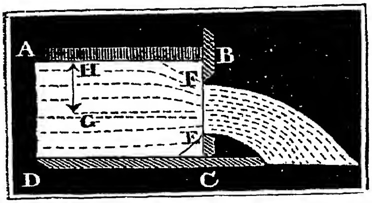
§ 321. *Efflux of Water in Motion.*—We have hitherto assumed that the head of water has been measured in still water; we must now, therefore, consider the case when only the head of water in motion, and flowing with a certain velocity towards the orifice, can be measured. Let us suppose the case of a rectangular lateral orifice, and represent its breadth by b, and the heads of water with respect to both horizontal edges h_1 and h_2, the height due to the velocity c of the affluent water by k, we shall then have the theoretical discharge:

$$Q = \frac{2}{3} b \sqrt{2g} [h_1 + k)^{\frac{3}{2}} - (h_2 + k)^{\frac{3}{2}}].$$

This formula is not directly applicable to the determination of the discharge, because the height due to the velocity:

$k = \frac{c^2}{2g} = \frac{1}{2g}\left(\frac{Q}{G}\right)^2$ is again dependent on Q, and further transformation leads to a complicated equation of a higher order, hence it is far simpler to put the effective discharge $Q_1 = \mu_1 ab \sqrt{2gh}$, and understand by μ_1 not the mere co-efficient of efflux, but one especially dependent on the ratios of the transverse sections. Most frequently, this case presents itself when the object is to measure water flowing

Fig. 433.



in canals and courses, because it is seldom possible in this case to dam up the water so high by a transverse section BC, Fig. 433, containing the orifice of discharge, that the orifice EF becomes only a small part, compared with the transverse section of the stream of water flowing to it; and, hence, the velocity of the last very small compared with the mean velocity.

From experiments made by the author on this subject with Poncelet orifices, where the head of water is measured one metre above the plane of the orifice, the expression: $\frac{\mu_n - \mu_0}{\mu_0} = 0,641 \left(\frac{F}{G}\right)^2 = 0,641 \, . \, n^2$, may be taken as tolerably accurate, when $n = \frac{F}{G}$ is the ratio of the transverse section, which, however, should not much exceed $\frac{1}{2}$; further, μ_0 represents the co-efficient for general contraction, taken from Poncelet's table corresponding to the present case. If b be the breadth, a the depth of the orifice, B the breadth and A the depth of the fluid stream, and h the depth of the upper side of the orifice below the surface of water, we have, accordingly, the effective discharge:

$$Q_1 = \left[1 + 0,641 \left(\frac{ab}{AB}\right)^2\right] \mu_0 \, . \, ab \sqrt{2g\left(h + \frac{a}{2}\right)}.$$

The following table serves for shortening the calculation in cases of application.

n	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50
$\frac{\mu_n - \mu_0}{\mu_0}$	0,002	0,006	0,014	0,026	0,040	0,058	0,079	0,103	0,130	0,160

Example. To find the quantity of water conducted through a course 3 feet broad, a board is placed across, with a 2 feet wide and 1 foot deep rectangular orifice, and the water in this way is so dammed up that it at last attains a height of $2\frac{1}{4}$ feet above the bottom, and $1\frac{3}{4}$ above the lower edge of the orifice. The theoretical discharge is $Q = ab \sqrt{2gh} = 1 \, . \, 2 \, . \, 8{,}02 \sqrt{1{,}25} = 16{,}04 \, . \, 1{,}118 = 17{,}93$ cubic feet; the co-efficient of efflux for perfect contraction may be put 0,689, and the ratio of the transverse sections $n = \frac{F}{G} = \frac{ab}{AB} = \frac{1 \, . \, 2}{2{,}25 \, . \, 3} = 0{,}296$; hence, it follows, that the co-efficient of efflux for the present ratio of discharge:
$= (1 + 0{,}641 \, . \, 0{,}296^2) \mu_0 = 1{,}056 \, . \, 0{,}602 = 0{,}6357$, and the effective quantity discharged $= 17{,}93 \, . \, 0{,}6357 = 11{,}31$ cubic feet.

§ 322. Imperfect contraction very often occurs in the efflux through wiers, as in Fig. 422. Wiers may take up a part only of the breadth of the reservoir or canal, or the whole breadth. In the latter case, contraction at the sides of the aperture does not take place, and for this reason more water flows through them than through wiers of the first kind. The author has made experiments also on these circum-

stances of efflux, and deduced from the results formulas by which the corresponding co-efficients may be estimated with tolerable certainty with the assistance of the ratio of the sections $n = \frac{G}{V} = \frac{h\,b}{A\,B}$. If we retain the denominations of the former paragraph, we then have for the Poncelet wiers:

$$\frac{\mu_n - \mu_0}{\mu_0} = 1{,}718 \left(\frac{F}{G}\right)^4 = 1{,}718 \, . \, n^4,$$

and for wiers occupying the entire breadth of the canal:

$$\frac{\mu_n - \mu_0}{\mu_0} = 0{,}041 + 0{,}3693\, n^2,$$

hence, in the first case, the discharge is:

$$Q_1 = \tfrac{2}{3}\left[1 + 1{,}718 \left(\frac{h\,b}{A\,B}\right)^4\right] \mu_0 \, . \, b \sqrt{2gh^3}.$$

And in the second:

$$Q_1 = \tfrac{2}{3}\left[1{,}041 + 0{,}3693 \left(\frac{h}{A}\right)^2\right] \mu_0 \, . \, b \sqrt{2gh^3},$$

where h represents the head of water EH above its sill F, measured at about 3 feet 6 inches back from the wier.

In the following tables, the corrections $\frac{\mu_n - \mu_0}{\mu_0}$, for the most simple values of n are put down.

TABLE I.

CORRECTIONS FOR THE PONCELET WIERS.

n	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50
$\frac{\mu_n - \mu_0}{\mu_0}$	0,000	0,000	0,001	0,003	0,007	0,014	0,026	0,044	0,070	0,107

TABLE II.

CORRECTIONS FOR WIERS OVER THE ENTIRE SIDE, OR WITHOUT ANY LATERAL CONTRACTION.

n	0,00	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50
$\frac{\mu_n - \mu_0}{\mu_0}$	0,041	0,042	0,045	0,049	0,056	0,064	0,074	0,086	0,100	0,116	0,133

Example. To determine the quantity of water carried off by a canal 5 feet broad, a waste board is applied, with an outward sloping edge, over which the water is allowed to flow after it has ceased to rise; the head of water above the bottom of the canal is $3\frac{1}{2}$ feet, and above the edge $1\frac{1}{2}$ feet, hence the theoretical discharge is $Q = \frac{2}{3} \, . \, 5 \, . \, 8{,}02$.

$\left(\frac{3}{2}\right)^{\frac{3}{2}} = 48,12$ cubic foot. The co-efficient of efflux is, since $\frac{h}{A} = \frac{1,5}{3,5} = \frac{3}{7}$ and $\mu_0 =$ 0,577,
$\mu_{\frac{3}{7}} = [1,041 + 0,3693 \cdot (\frac{3}{7})^2] \cdot 0,577 = 1,110 \cdot 0,577 = 0,64$, hence the effective discharge $Q_1 = 0,64 \cdot Q = 0,64 \cdot 48,12 = 30,79$ cubic feet.

CHAPTER III.

ON THE EFFLUX OF WATER THROUGH TUBES.

§ 323. *Short Tubes, or Mouth-pieces.*—If water is allowed to flow through *short tubes*, or *mouth-pieces*, other relations take place than when it flows through orifices in a thin plate, or through outwardly sloping orifices in a thick plate. When the tube is prismatic, and its length $2\frac{1}{2}$ to 3 times that of its width, it then gives an uncontracted and opaque stream, which has a small distance of projection, and hence, also, a smaller velocity than that of a jet flowing, under otherwise similar circumstances, through an orifice in a thin plate. If, therefore, the tube KL has the same transverse section as the orifice F, Fig. 434; and if also the head of water of both is one and the same, we then obtain in LR a troubled and uncontracted, and, therefore, a thicker jet, and in FH a clear and contracted, and, therefore, thinner one; and, it may be observed, that the distance of the projection ER is less than that of DH. This ratio of efflux only takes

Fig. 434.

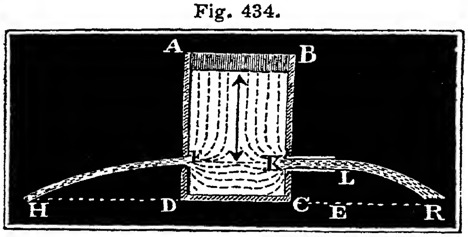
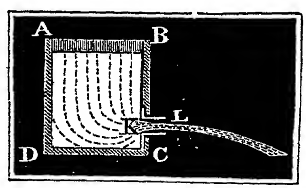


Fig. 435.

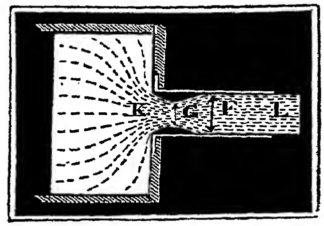


place when the tube is of a given length; if it is shorter, or scarcely as long as it is broad, then the jet KL, Fig. 435, will not touch the sides of the tube, the tube will have no influence on the efflux, and the jet will be the same as through orifices in a thin plate.

Sometimes in tubes of greater length, the fluid stream does not entirely fill the tube, namely: when the water is not allowed to come into contact with the sides of the tube; but if in this case we close the outer orifice by the hand or by a board for a few moments, a stream will then be formed which will entirely fill the tube, and the so-called *full flow* will then take place. Contraction of the fluid vein takes place also in the flow through tubes, but the place of contrac-

tion is here in the interior of the tube. We may be convinced of this, if we avail ourselves of glass tubes, such as KL, Fig. 436, and color the water, for in this case we shall remark, that there is progressive motion only in the middle of the transverse section G close behind the place of entrance K, but not at the outside of it, and that it is a sort of eddying motion which takes place. But it is the capillarity, or the adhesion of the water to the sides of the tube, which causes the fluid entirely to fill the end FL of the tube. The water flowing from the tube has only a pressure equal to that of the atmosphere, but the contracted section G is only a times the size of the section F of the tube, and for this reason the velocity in it $\frac{1}{a}$ times as great as the velocity of efflux v; hence the pressure of the water in the vicinity of G is

Fig. 436.



$\frac{\left(\frac{1}{a} v\right)^2}{2g} - \frac{v^2}{2g} = \left[\left(\frac{1}{a}\right)^2 - 1\right]\frac{v^2}{2g}$ (§ 307) less than at its exit, or than the atmospheric pressure. If we bore a narrow hole in the tube at G, no discharge will pass through it, but there will be an absorption of air rather; the full discharge and the action of the tube will at last entirely cease if the hole be made wider, or more holes bored.

§ 323. *Cylindrical Tubes.*—Numerous experiments have been made on the flow of water through cylindrical additional tubes; but the results vary considerably from one another.* The co-efficients of Bossut are those, which from their smallness (0,785) have been found to vary most from others. From the experiments of Michelotti, with tubes from $\frac{1}{2}$ to 3 inches width, and with a head of water of from 3 to 20 feet, the mean of this co-efficient is: $\mu = 0{,}813$. The experiments of Bidone, Eytelwein and d'Aubuisson vary very little from this. The mean, however, which may be adopted, and which corresponds particularly with the author's experiments on the discharge through short mouth-pieces $= 0{,}815$. As we have found this for orifices in a thin plate 0,615, it follows that, under otherwise similar circumstances and relations, $\frac{815}{615} = 1{,}325$ times as much water flows through cylindrical additional tubes, as through round orifices in a thin plate. These co-efficients, moreover, increase as the width of tubes becomes less, and but slightly with the increase of the head of water or velocity of efflux. According to the author's experiments, under a pressure of from 9 to 24 inches for tubes three times as long as broad:

* A considerable series of experiments on the flow of water through adjutages was, some years since, performed by a committee of the Franklin Institute, which yet await a proper reduction to render the results available.—Am. Ed.

at	1 or ,4 in.	2 or ,8 in.	3 or 1,2	4 centimetres width, or 1,6 inches width.
$\mu =$	0,843	0,832	0,821	0,810

According to this table, therefore, the co-efficients increase considerably as the width of the tubes decreases. Buff found for tubes 2,79 lines wide, and 4,3 lines long, the co-efficients of efflux gradually to increase from 0,825 to 0,855, when the head of water sank from 33 to $1\frac{1}{2}$ inches successively.

The author found a co-efficient of efflux of 0,819 for the flow of water through rectangular additional tubes.

If the additional tubes *KL*, Fig. 437, are on the inside partially confined; if, for instance, one side is contiguous to the bottom, and if a partial contraction is produced thereby, then the co-efficient of efflux, from the author's experiments, does not perceptibly increase, but the

Fig. 437.

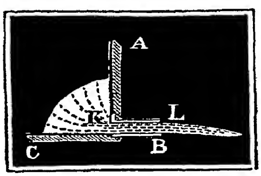
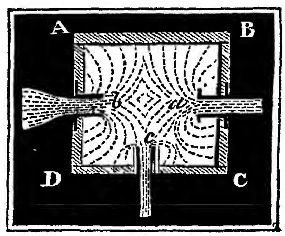


Fig. 438.



water flows away at different parts of the section, with different velocities, and, of course, from the side *BC* faster than from the side opposite to it. If the inner anterior surface of a tube does not coincide with the side surface, but projects, as *a*, *b*, *c*, Fig. 438, then this tube is called an *internal additional tube*. If the anterior surface of this tube is at least $\frac{1}{5}$th as broad as the tube is wide, as for example *a*, then the co-efficient of efflux will remain the same as if this surface were in the plane of the side, but if the anterior surface be less, as *b*, *c*, the co-efficient will then be less. For a very small and almost vanishing anterior surface, according to the experiments of Bidone and the author, this amounts to 0,71 if the vein fills the tube, and 0,53 (compare § 318) if it does not quite fill the inner sides of the tube. In the first case (*b*) the fluid stream is broken and divergent, like a brush; and in the second (*c*) strongly contracted and quite crystalline.

§ 324. *Co-efficient of Resistance.*—As water flows without contraction from prismatic additional tubes, it follows that, in its efflux through these mouth-pieces, the co-efficient of contraction = unity, and the co-efficient of velocity φ = the co-efficient of efflux μ. A discharge Q with the velocity v, possesses a *vis viva* $\frac{Q\gamma}{g}v^2$, and is ca-

pable of producing the mechanical effect $\frac{v^2}{2g} Q \gamma$ (§ 71). But now the theoretical velocity of efflux $= \frac{v}{\phi}$, hence the mechanical effect $\frac{1}{\phi^2} \cdot \frac{v^2}{2g} \cdot Q \gamma$ corresponds to the mass of water flowing out, and the discharge Q accordingly loses by efflux the mechanical effect

$$\left(\frac{1}{\phi^2} \cdot \frac{v^2}{2g} - \frac{v^2}{2g}\right) Q \gamma = \left(\frac{1}{\phi^2} - 1\right) \frac{v^2}{2g} Q \gamma.$$

For efflux through orifices in a thin plate, the mean of $\phi = 0,97$, hence the loss of effect here amounts to

$$\left[\left(\frac{1}{0,97}\right)^2 - 1\right] \frac{v^2}{2g} Q \gamma = 0,063 \frac{v^2}{2g} Q \gamma;$$

for efflux through short cylindrical tubes $\phi = 0,815$, and the corresponding loss of effect

$$= \left[\left(\frac{1}{0,815}\right)^2 - 1\right] \frac{v^2}{2g} Q \gamma = 0,505 \frac{v^2}{2g} Q \gamma,$$

i. e. eight times as great as for efflux through orifices in a thin plate. In rendering available the *vis viva* of flowing water, it is consequently better to let the fluid flow through orifices in a thin plate, than through prismatic tubes. But if the inner edges in which the tube meets the sides of the cistern are rounded, and by this a gradual passage from the cisterns into the tube effected, the co-efficient of efflux will then rise to 0,96, and the loss of mechanical effect will be brought down to 8½ per cent. In shorter adjutages, accurately rounded, having the form of the contracted fluid vein $\mu = \phi = 0,97$, and hence the loss of mechanical effect as for orifices in a thin plate = 6 per cent.

A head of water $\left(\frac{1}{\phi^2} - 1\right) \frac{v^2}{2g} Q \gamma$ is due to the loss of mechanical effect $\left(\frac{1}{\phi^2} - 1\right) \frac{v^2}{2g}$; hence we may also suppose that from the obstacles to the efflux, the head of water suffers the loss $\left(\frac{1}{\phi^2} - 1\right) \frac{v^2}{2g}$, and assume after deduction of this loss, that the residuary part of the head of water is expended in generating the velocity. We may call this loss $\left(\frac{1}{\phi^2} - 1\right) \frac{v^2}{2g}$, which increases with the square of the velocity of efflux, the *height due to the resistance*, and the co-efficient $\frac{1}{\phi^2} - 1$, with which the height due to the velocity is to be multiplied to obtain the height due to the resistance, the *co-efficient of resistance*. We shall represent in what follows, the co-efficient, expressing the ratio of the height of resistance to the head of water, by the letter ζ; therefore, the height due to the resistance itself may be expressed by $\zeta \cdot \frac{v^2}{2g}$. By the formula $\zeta = \frac{1}{\phi^2} - 1$ and $\phi = \frac{1}{\sqrt{1+\zeta}}$,

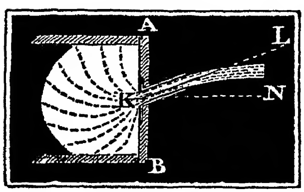
the co-efficient of resistance may be calculated from the co-efficient of velocity, and *vice versâ*.

Examples.—1. What discharge will flow through a 2 inch wide tube, under a head of water of 3 feet, which corresponds to a co-efficient of resistance $\zeta = 0,4$? $\phi = \frac{1}{\sqrt{1,4}} = 0,845$; hence, $v = 0,845 \,.\, 8,02 \sqrt{3} = 12,05$ feet; further, $F = \left(\frac{1}{12}\right)^2 \pi = 0,02182$ square feet; consequently, the quantity of water sought is $Q = 0,263$ cubic feet.—2. If a tube of 2 inches width, under a pressure of 2 feet, deliver in a minute 10 cubic feet of water, its co-efficient of efflux, or of velocity, is then $\phi = \frac{Q}{F\sqrt{2\,g\,h}}$ $= \frac{10}{60 \,.\, 0,02182 \,.\, 8,02 \,.\, \sqrt{2}} = \frac{1}{1,050\sqrt{2}} = 0,674$, the co-efficient of resistance $= \left(\frac{1}{0,674}\right)^2 - 1 = 1,201$; and lastly, the loss in head of water produced by the resistances of the tube:

$$= 1,201 \frac{v^2}{2g} = 1,201 \,.\, 0,0155 \left(\frac{Q}{F}\right)^2 = 0,0186 \,.\, \frac{1}{0,1309^2} = 1,085 \text{ feet.}$$

§ 325. *Oblique Additional Tubes.*—Obliquely attached or obliquely cut tubes give a smaller quantity of water than rectangularly attached, or rectangularly cut additional tubes, because the direction of the water in them becomes changed. Experiments conducted upon an extensive scale, have led the author to the following. If δ be the angle which the axis of the tube KL, Fig. 439, makes with the normal KN to the plane AB of the inner orifice of the tube; and if ζ be the co-efficient of resistance for rectangularly cut tubes, we shall then have the co-efficient of resistance for the inclined tube: $\zeta_1 = \zeta + 0,303\, sin.\, \delta + 0,226\, sin.\, \delta^2$. Let us take for ζ the mean value 0,505, and we shall obtain:

Fig. 439.



for δ^0 =	0	10	20	30	40	50	60°
The co-efficient of resistance ζ_1 =	0,505	0,565	0,635	0,713	0,794	0,870	0,937
The co-efficient of efflux μ_1 =	0,815	0,799	0,782	0,764	0,747	0,731	0,719

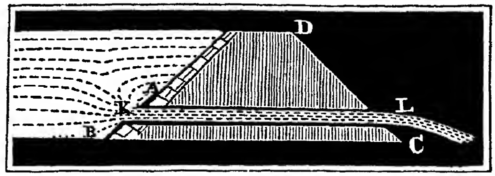
From this, for example, the co-efficient of resistance of an additional tube deviating by 20° from the axis is $\zeta = 0,635$, and the co-efficient of efflux $= \frac{1}{\sqrt{1,635}} = 0,782$, and for 35° deviation, the first $= 0,753$, and the last $= 0,755$.

In general, these inclined and additional tubes are larger than we have hitherto assumed, and they should be longer too, because the water would not otherwise perfectly fill the tube. The preceding formula represents only that part of the resistance which is due to

that portion of tube at the inner orifice, which is three times as long as the tube is wide. The resistance which the remaining portion of tube opposes to the motion of the water, will be given subsequently.

Example. If the plane of the inner orifice AB of a horizontally lying pond sluice KL, Fig. 440, as likewise the interior surface of the pond dam, is inclined 40° to the horizon, then the axis of the pipe makes, with the normal to this plane, an angle of 50°, and hence the co-efficient of resistance for efflux through the portion of the interior orifice of this tube is $= 0{,}870$; and if now the co-efficient of resistance 0,650 were due to the remaining and longer portion, the co-efficient of resistance of the entire tube would then be $= 0{,}870 + 0{,}650 = 1{,}520$, and hence the co-efficient of efflux $= \frac{1}{\sqrt{1 + 1{,}520}} = \frac{1}{\sqrt{2{,}520}} = 0{,}630$. For a 10 feet head of water and 1 foot width of tube, the following discharge would be given:

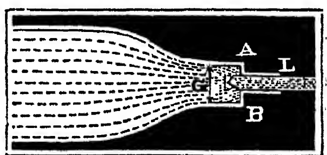
Fig. 440.



$$Q = 0{,}63 \cdot \frac{\pi}{4} \cdot 8{,}02 \sqrt{10} = 12{,}55 \text{ cubic feet.}$$

§ 326. *Imperfect Contraction.*—When a cylindrical additional tube KL, Fig. 441, is inserted into a plane wall AB, whose area G does not much exceed the transverse section F of the tube, the water then comes to the place of insertion with a velocity which must not be disregarded, and it then issues into the tube with imperfect contraction only, on which account the velocity of efflux is again greater than when the water before entrance into the tube is to be assumed as still. Again if $\frac{F}{G} = n$ is the ratio of the section of the tube to that of the area of the side, and further, μ_0 be the co-efficient of efflux for perfect contraction, where $\frac{F}{G}$ may be equated to 0, we shall have, according to the experiments of the author, to put the co-efficient of efflux for imperfect contraction, or for the ratio of the sections n:

Fig. 441.



$$\frac{\mu_n - \mu_0}{\mu_0} = 0{,}102\, n + 0{,}067\, n^2 + 0{,}046\, n^3 \text{ or}$$

$$\mu_n = \mu_0 (1 + 0{,}102\, n + 0{,}067\, n^2 + 0{,}046\, n^3).$$

If the transverse section of the tube occupies the sixth part of the whole surface of the side, there is:

$$\mu_{\frac{1}{6}} = \mu_0 (1 + 0{,}102 \cdot \tfrac{1}{6} + 0{,}067 \cdot \tfrac{1}{36} + 0{,}046 \cdot \tfrac{1}{216})$$
$$= \mu_0 (1 + 0{,}017 + 0{,}0019 + 0{,}0002) = 1{,}019\, \mu_0, \text{ or}$$

μ_0 being put $= 0{,}815$, $\mu_{\frac{1}{6}} = 0{,}815 \cdot 1{,}019 = 0{,}830$.

The following useful and convenient table gives somewhat more accurately the values for correction $\frac{\mu_n - \mu_0}{\mu_0}$

TABLES

OF CORRECTION FOR IMPERFECT CONTRACTION, BY EFFLUX THROUGH SHORT CYLINDRICAL TUBES.

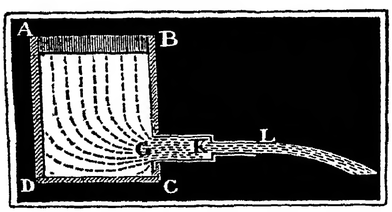
n	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50
$\frac{\mu_n - \mu_0}{\mu_0}$	0,006	0,013	0,020	0,027	0,035	0,043	0,052	0,060	0,070	0,080

n	0,55	0,60	0,65	0,70	0,75	0,80	0,85	0,90	0,95	1,00
$\frac{\mu_n - \mu_0}{\mu_0}$	0,090	0,102	0,114	0,127	0,138	0,152	0,166	0,181	0,198	0,227

By efflux through short parallelopipedical tubes, these corrections are nearly the same.

These co-efficients are especially applicable to the efflux of water through compound tubes; for example, in the case represented in Fig. 442, where the orifice of the short tube KL enters the wider short tube CK, whose orifice again lies in the cistern AC. Imperfect contraction takes place at the entrance of the water from the wider into the narrower tube, and hence the co-efficient of efflux must be determined by the last rule. If we put the co-efficient of resistance corresponding to the co-efficient of efflux $= \zeta_1$, the co-efficient of resistance for the entrance into the wider tube from the cistern $= \zeta$, the head of water $= h$, the velocity of efflux $= v$, and the ratio $\frac{F}{G}$ of the section of the tubes $= n$, therefore, the velocity of the water in the wider tube $= nv$, then the formula gives:

Fig. 442.



$$h = \frac{v^2}{2g} + \zeta \cdot \frac{(nv)^2}{2g} + \zeta_1 \cdot \frac{v^2}{2g}, \text{ i. e. } h = (1 + n_2 \zeta + \zeta_1) \frac{v^2}{2g}.$$

And hence:

$$v = \frac{\sqrt{2gh}}{\sqrt{1 + n^2 \zeta + \zeta_1}}.$$

Example. What discharge will the apparatus delineated in Fig. 442 deliver, if the head of water $h = 4$ feet, the width of the narrower tube 2 inches, and that of the wider one 3 inches? $n = (\frac{2}{3})^2 = \frac{4}{9}$, hence $\mu_{\frac{4}{9}} = 1{,}069 \cdot 0{,}815 = 0{,}871$, and the corresponding co-efficient of resistance $\zeta_1 = \left(\frac{1}{0{,}871}\right)^2 - 1 = 0{,}318$; but now we have $\zeta = 0{,}505$ and $n^2 \cdot \zeta = \frac{16}{81} \cdot 0{,}505 = 0{,}099$; hence it follows, that $1 + n^2 \zeta + \zeta_1$

$= 1 + 0,099 + 0,318 = 1,417$, and the velocity of efflux $v = \frac{8,02 \,.\, \sqrt{4}}{\sqrt{1,417}} = \frac{16,04}{\sqrt{1,417}}$ $= 14,34$ feet. Again, since the transverse section of the tube $= 0,02182$ square feet, the discharge will be $Q = 14,34 \,.\, 0,02182 = 0,313$ cubic feet.

§ 327. *Conical Tubes.*—Additional conical tubes give a discharge different from that of prismatic or cylindrical tubes; they are either conically convergent, or conically divergent; in the first case the outer orifice is smaller, and in the second case larger than the inner orifice. The co-efficients of efflux for the first tubes are greater, and for the last, less than for cylindrical tubes. One and the same conical tube gives more water when the wider orifice is made the exit orifice, as K in Fig. 443, than when it is turned inwards, as L in the same figure, except that it does not give a greater quantity in proportion as the wider orifice exceeds the narrower. If many, as Venturi and Eytelwein, give for conically divergent tubes, a greater co-effi-

Fig. 443.

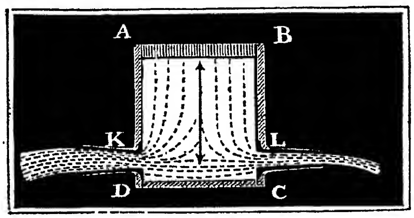
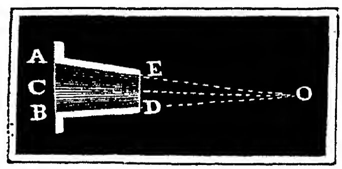
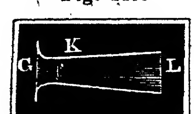


Fig. 444.



cient of efflux, than for conically convergent tubes, it must always be borne in mind, that they take the narrower transverse section for the orifice. The following experiments instituted at pressures of from 0,25 to 3,3 metres, with a tube AD 9 centimetres long, Fig. 444, bring before us the effect of conicalness in tubes. The width of these tubes at one extremity amounted to $DE = 2,468$, at the other $AB = 3,228$ centimetres, and the angle of convergence, *i. e.* the angle AOB, which the oppositely situated sides AE and BD of a section in the direction of the longer axis include $= 4^\circ, 50'$. By efflux through the narrower orifice, the co-efficient was $= 0,920$, but by efflux through the wider it was $= 0,553$, and if in the calculation, we take the narrower entrance orifice for the transverse section, it will give $= 0,946$. In the first case, when the tube was applied as a conically convergent adjutage, the fluid vein was little contracted, thick and smooth; but, in the second case, when the tube served as a conically divergent adjutage, it was strongly divergent, broken, and spouting. Venturi and Eytelwein have experimented further on efflux through conically divergent tubes. Both philosophers have applied these conical tubes to cylindrical and conoidal adjutages, made after the form of the contracted fluid vein. By such a connection as is represented in Fig. 445, where the divergent portion KL of the outer orifice is between 12 and $21\frac{1}{2}$ lines wide, and $8\frac{1}{1}\frac{3}{8}$ of an inch long, and the angle

Fig. 445.



of convergence estimated at 5°, 9′. Eytelwein found $\mu = 1,5526$ when he took the narrow end for the orifice, and, on the other hand, $\mu = 0,483$ for the wider end, in which he was right. Through this combined adjutage there certainly flows $\frac{1,5526}{0,615} = 2,5$ times as much as through a simple orifice in a thin plate, and $\frac{1,5526}{0,815} = 1,9$ times as much as through a short cylindrical tube. With small velocities and greater divergence, it is scarcely possible, even by previously closing the tubes, to bring about a full flow.

§ 328. The most ample experiments have been made by d'Aubuisson and Castel on efflux through conically convergent additional tubes. The tubes for this purpose were of great variety, of different lengths, widths, and angles of convergence. The most extensive experiments were those made with tubes of 1,55 centimetres width at the discharging orifice, and of from 2,6 times greater, *i. e.* of 4 centimetres in length, for which reason we will here communicate the results in the following table. The head of water was, throughout, 3 metres. The discharges were measured by a special gauge-cistern; but in order to obtain besides the co-efficients of efflux, those of the velocity and contraction, the amplitude of the jet, due to given heights, were measured, and from these the velocity of efflux (*see* § 38, *Ex.* 2) calculated. The ratio $\frac{v}{\sqrt{2gh}}$ of the effective velocity v to the theoretical $\sqrt{2\,g\,h}$ gave the co-efficient of velocity ϕ, as also the ratio $\frac{Q}{F.\sqrt{2\,g\,h}}$ of the effective discharge Q to the theoretical $F\sqrt{2\,g\,h}$ gave the co-efficient of efflux μ, and, lastly, the ratio of both co-efficients, *i. e.* $\frac{\mu}{\phi}$, determined the co-efficient of contraction a.

TABLE

OF THE CO-EFFICIENTS OF EFFLUX AND VELOCITY FOR EFFLUX THROUGH CONICALLY CONVERGENT TUBES.

Angle of convergence	Co-efficient of efflux.	Co-efficient of velocity.	Angle of convergence	Co-efficient of efflux.	Co-efficient of velocity.
0° 0′	0,829	0,829	13° 24′	0,946	0,963
1° 36′	0,866	0,867	14° 28′	0,941	0,966
3° 10′	0,895	0,894	16° 36′	0,938	0,971
4° 10′	0,912	0,910	19° 28′	0,924	0,970
5° 26′	0,924	0,919	21° 0′	0,919	0,972
7° 52′	0,930	0,932	23° 0′	0,914	0,974
8° 58′	0,934	0,942	29° 58′	0,895	0,975
10° 20′	0,938	0,951	40° 20′	0,870	0,980
12° 4′	0,942	0,955	48° 00′	0,847	0,984

From this table it is seen that the co-efficients of efflux attain their maximum 0,946 for a tube of $13\frac{1}{2}^{\circ}$ lateral convergence; that, on the other hand, the co-efficients of velocity come out always greater and greater, the greater is the angle of convergence. The following example will show how this table may be used in those cases which present themselves in practice.

Example.—What discharge will a short conoidal tube of $1\frac{1}{2}$ inches width at the outer orifice, and of 10° convergence, deliver under a pressure of 16 feet? According to the experiments of the author, a cylindrical tube of this width gives $\mu = 0,810$; d'Aubuisson's tube, however, gave $\mu = 0,829$, therefore about $0,829 - 0,810 = 0,019$ more. Now from the table for a tube of 10° convergence, μ is $= 0,937$; hence, for the given tube we may put $\mu = 0,937 - 0,019 = 0,918$, whence the discharge

$$Q = 0,918 \cdot \frac{\pi}{4 \cdot 8^2} \cdot 8,02 \sqrt{4} = \frac{8,02 \cdot 0,918\,\pi}{64} = 0,3614 \text{ cubic feet.}$$

§ 329. *Resistance of Friction.*—The longer prismatic or cylindrical tubes are, the more they retard the efflux; hence we must assume that the sides of the tubes offer resistance to the motion of the water by the friction, adhesion, or viscidity of the fluid acting against them. From reason and from numerous observations and measurements, we may assume that this resistance of friction is independent of the pressure, that it increases directly as the length l, and inversely as the width d, and, therefore, proportional to the ratio $\frac{l}{d}$. Moreover, it appears that this resistance is greater for great and less for less velocities, and that it very nearly increases with the square of the velocity v of the water. If we measure this resistance by the height of a column of water, which must be deducted from the entire head h, in order to obtain the height requisite for the generation of the velocity, we may then put for this height, which we shall term *the height due to the resistance*, $h_1 = \zeta_1 \cdot \frac{l}{d} \cdot \frac{v^2}{2g}$, where ζ_1 represents a number, from experiment, which we may call the *height due to the resistance of friction.* There is a greater loss, therefore, of pressure or head of water from the friction of the water in the tube, the greater the ratio $\frac{l}{d}$ of the length to the width is, and the greater the height due to the velocity $\frac{v^2}{2g}$. From the discharge Q and the transverse section of the tube $F = \frac{\pi d^2}{4}$ there follows the velocity $v = \frac{4\,Q}{\pi d^2}$, and hence the height due to the friction:

$$h_1 = \zeta_1 \cdot \frac{l}{d} \cdot \frac{1}{2g} \left(\frac{4\,Q}{\pi d^2}\right)^2 = \zeta_1 \cdot \frac{1}{2g} \cdot \left(\frac{4}{\pi}\right)^2 \cdot \frac{l\,Q^2}{d^5}.$$

In order to obtain the least possible loss of head of water, or fall, in leading a certain quantity of water Q, the pipe must be made as wide as possible, and not unnecessarily long. A double width requires, for instance, only $(\frac{1}{2})^5 = \frac{1}{32}$ of the fall that the single width does.

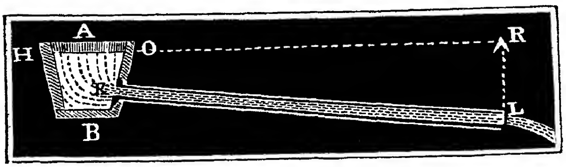
If the transverse section of a tube be rectangular, and of the depth a and the breadth b, we must substitute for

$$\frac{1}{d} = \frac{1}{4} \cdot \frac{\pi d}{\frac{1}{4}\pi d^2} = \frac{1}{4} \cdot \frac{\text{circumference}}{\text{area}} = \frac{1}{4} \cdot \frac{2(a+b)}{ab} = \frac{a+b}{2ab},$$

whence it follows: $h_1 = \zeta \cdot \frac{l(a+b)}{2ab} \cdot \frac{v^2}{2g}$.

By means of this formula for the resistance of friction in pipes, we may also find the velocity and the quantity of efflux which a pipe, of given length and width, will conduct under a given pressure. It is quite the same, whether the tube KL, Fig. 446, is horizontal, inclines, or ascends, if only by the head of water is understood the depth RL of the middle point L of the orifice of the tube below the surface of water HO of the efflux reservoir. If h is the head of water, h_1 the height due to the resistance for the orifice of entrance, and h_2 that for the remaining portion of the tube, we then have:

Fig. 446.



$h-(h_1+h_2) = \frac{v^2}{2g}$, or $h = h_1 + h_2 + \frac{v^2}{2g}$. If ζ represents the co-efficient of resistance for the portion of tube next the cistern, and ζ_1 the co-efficient of the resistance of friction of the rest of the tube, we then have

$$h = \zeta \cdot \frac{v^2}{2g} + \zeta_1 \cdot \frac{l}{d} \cdot \frac{v^2}{2g} + \frac{v^2}{2g},$$

$$\text{or, 1.) } h = \left(1 + \zeta + \zeta_1 \frac{l}{d}\right) \frac{v^2}{2g},$$

$$\text{and 2.) } v = \frac{\sqrt{2gh}}{\sqrt{1 + \zeta + \zeta_1 \cdot \frac{l}{d}}}.$$

From the last formula the discharge $Q = Fv$ is given.

For very long tubes $1 + \zeta$ is small compared with $\zeta_1 \frac{l}{d}$, whence, simply, $h = \zeta_1 \frac{l}{d} \cdot \frac{v^2}{2g}$, and inversely,

$$v = \sqrt{\frac{1}{\zeta_1} \cdot \frac{d}{l} \cdot 2gh}.$$

§ 330. The co-efficient of friction, like the co-efficient of efflux, is not quite constant; it is greater for small, and less for great velocities; *i. e.* the resistance of the friction of water in tubes does not increase exactly with the square of the velocity, but with some other power of it. Prony and Eytelwein have assumed, that the head of water lost by the resistance due to friction ought to increase as the

simple velocity and as its square, and have given for it the expression $h_1 (\alpha v + \beta v^2) \frac{l}{d}$, where α and β are co-efficients deduced from experiment. To determine these co-efficients, 51 experiments, which, at various times, were made by Couplet, Bossut, and Du Buat, on the motion of water through long tubes, were made use of by these hydraulicians.

Prony found from this, that $h_1 = (0{,}0000693\, v + 0{,}0013932\, v^2) \frac{l}{d}$, Eytelwein $h_1 = (0{,}0000894\, v + 0{,}0011213\, v^2) \frac{l}{d}$; d'Aubuisson assumes $h_1 = (0{,}0000753\, v + 0{,}001370\, v^2) \frac{l}{d}$ metres.

A formula, discovered by the author, agrees more accurately with observation. It has the form

$$h_1 = \left(\alpha + \frac{\beta}{\sqrt{v}}\right) \frac{l}{d} \frac{v^2}{2g},$$

and is based on the hypothesis, that the resistance of friction increases simultaneously as the square, and as the square root of the cube of the velocity. From this we have the co-efficient of resistance $\zeta_1 = \alpha + \frac{\beta}{\sqrt{v}}$, and the height due to the resistance of friction $h_1 = \zeta_1 \cdot \frac{l}{d} \frac{v^2}{2g}$.

For the measurement of the co-efficient of resistance ζ_1, or of the auxiliary constants α and β, not only the determinations of Prony and Eytelwein from the 51 experiments of Couplet, Bossut, and Du Buat were used by the author, but also 11 experiments made by him, and 1 experiment by Gueymard in Grenoble. The older experiments extend only to velocities of from 0,043 to 1,930 metres; in the experiments of the author, however, the extreme limit of velocity reached to 4,648 metres. The widths of the tubes, in the older experiments, were 27 mm. = 1.06 in.; 36 mm. = 1.95 in.; 54 mm. = 2.12 in.; 135 mm. = 5.31 in.; and 490 mm. = 19.29 in.; later experiments were conducted with tubes of 33 mm. = 1.29 in.; 71 mm. = 2.79 in.; and 275 mm. = 5.31 in. By means of the method of least squares, it has been found from the 63 experiments laid down:

$$\zeta_1 = 0{,}01439 + \frac{0{,}0094711}{\sqrt{v}}; \text{ therefore,}$$

$$h_1 = \left(0{,}01439 + \frac{0{,}0094711}{\sqrt{v}}\right) \frac{l}{d} \cdot \frac{v^2}{2g} \text{ metre;}$$

for Prussian measure:

$$h_1 = \left(0{,}01439 + \frac{0{,}016921}{\sqrt{v}}\right) \frac{l}{d} \cdot \frac{v^2}{2g} \text{ feet,}$$

or for English measure:

$$h_1 = \left(0{,}01439 + \frac{0{,}017963}{\sqrt{v}}\right) \frac{l}{d} \cdot \frac{v^2}{2g} \text{ feet.}$$

§ 331. For facilitating the calculation, the following table of the co-efficients of resistance has been compiled. We see from this, that

the variability of these co-efficients is not inconsiderable, as for 0,1 metre velocity it is = 0,0443, for 1 metre = 0,0239, and for 5 metres = 0,0186.

TABLE OF THE CO-EFFICIENTS OF FRICTION.*

v ft	v	10ths of a metre. 0	1 or 4 in.	2 or 8 in.	3 12 in.	4 16 in.	5 20 in.	6 24 in.	7 28 in.	8 32 in.	9 36 in.
ft. in.	Whole metres.										
0	0	∞	0,0443	0,0356	0,0317	0,0294	0,0278	0,0266	0,0257	0,0250	0,0244
3.4	1	0239	0234	0230	0227	0224	0221	0219	0217	0215	0213
6.7	2	0211	0209	0208	0206	0205	0204	0203	0202	0201	0200
9.10	3	0199	0198	0197	0196	0195	0195	0194	0193	0193	0192
13.0	4	0191	0191	0190	0190	0189	0189	0188	0188	0187	0187

We find in this table the co-efficients of resistance due to a certain velocity, when we look for the whole metre in the vertical, and the tenths in the first horizontal column, then proceed from the first number horizontally, and from the last vertically to the place where both motions meet; for example, for v = 1,3 metre ζ_1 = 0,0227, for v = 2,8, ζ_1 = 0,0201.

For the Prussian measure we may put:

v	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9 ft.
ζ_1	0,0679	0,0522	0,0453	0,0411	0,0383	0,0362	0,0346	0,0333	0,0322

v	1	$1\frac{1}{4}$	1	2	3	4	6	8	12	20 ft.
ζ_1	0,0313	0,0296	0,0282	0,0263	0,0242	0,0229	0,0213	0,0204	0,0192	0,0182

§ 332. *Long Tubes.*—With respect to the motion of water in long tubes or conducting pipes, the three following fundamental problems present themselves for solution.

1. The length l and the width d of the tube and the quantity of water Q to be conducted are given, and the head of water is required. We have first to calculate the velocity $v = \frac{Q}{F} = \frac{4\,Q}{\pi\, d^2} = 1,2732 \,.\, \frac{Q}{d^2}$, then to look for the co-efficient of friction ζ_1 corresponding to this value, in one of the last tables, and, lastly, to substitute the values of

* To apply this table to English measures, the velocity, if below 40 inches, will be taken out from the numbers under "10ths of a metre," and if above 36, the number of feet and inches in the left hand column will be used with the inches at the head. Thus, for v = 7 feet 11 inches we have on a line with 6 feet 7 inches at the left, and under 16 inches at top, the co-efficient, 0205.—Am. Ed.

d, l, v, ζ and ζ_1, where ζ represents the co-efficient of friction for the portion of the interior orifice, in the last formula

$$h = \left(1 + \zeta + \zeta_1 \frac{l}{d}\right) \frac{v^2}{2g}.$$

2. The length and width of the tube, as well as the head of water or fall, are given to determine the discharge. We must here find the velocity by the formula

$$v = \frac{\sqrt{2gh}}{\sqrt{1 + \zeta + \zeta_1 \cdot \frac{l}{d}}};$$

but since the co-efficient of resistance is not quite constant, but varies somewhat with v, we must know v approximately beforehand, in order to be able to find out ζ_1.

From v it then follows that $Q = \frac{\pi d^2}{4} v = 0,7854\, d^2 v$.

3. The discharge, the head of water, and the length of the tube are given, to determine the requisite width of the tube.

As $v = \frac{4\,Q}{\pi d^2}$, therefore $v^2 = \left(\frac{4\,Q}{\pi}\right)^2 \cdot \frac{1}{d^4}$,

we then have $2\,gh = \left(1 + \zeta + \zeta_1 \frac{l}{d}\right) \left(\frac{4\,Q}{\pi}\right)^2 \cdot \frac{1}{d^4}$, or

$2\,g\,h \cdot \left(\frac{\pi}{4\,Q}\right)^2 = (1 + \zeta) \frac{1}{d^4} + \zeta_1 \frac{l}{d^5}$, or

$2\,g\,h \cdot \left(\frac{\pi}{4\,Q}\right)^2 d^5 = (1 + \zeta)\, d + \zeta_1\, l$, hence the width of the tube

$$d = \sqrt[5]{\frac{(1 + \zeta)\, d + \zeta_1\, l}{2\,g\,h} \cdot \left(\frac{4\,Q}{\pi}\right)^2}.$$

But now $\left(\frac{4}{\pi}\right)^2 = 1,6212$, $1 + \zeta$ as a mean $= 1,505$ and $\frac{1}{2g} = 0,0155$, hence we may put:

$$d = 0,4787 \sqrt[5]{(1,505 \cdot d + \zeta_1\, l)\, \frac{Q^2}{h}} \text{ feet.}$$

This formula can only be used as a formula of approximation, because the unknown quantity d, and also the co-efficient ζ_1, dependent on $v = \frac{4\,Q}{\pi\, d^2}$, appear in it.

Examples.—1. What head of water does a conducting pipe, of 150 feet length and 5 inches width, require, if it is to carry off 25 cubic feet of water per minute? Here $v = 1,2732 \cdot \frac{25 \cdot 12^2}{60 \cdot 5^2} = 3,056$ feet, hence we may put $\zeta_1 = 0,0242$; and the head of water or the entire fall of the pipe will be:

$$h = \left(1,505 + 0,0242 \cdot \frac{150 \cdot 12}{5}\right) \cdot 0,0155 \cdot 3,056^2$$
$$= (1,505 + 8,712) \cdot 0,0155 \cdot 9,339 = 1,479 \text{ feet, (English.)}$$

2. What discharge will a conducting pipe, 48 feet long and 2 inches wide, with a 5 feet head of water, deliver? It will be:

$$v = \frac{8{,}02\sqrt{5}}{\sqrt{1{,}505 + \zeta_1 \frac{48 \cdot 12}{2}}} = \frac{17{,}88}{\sqrt{1{,}505 + 288 \cdot \zeta_1}}.$$

If we previously take $\zeta_1 = 0{,}020$, we shall obtain $v = \frac{17{,}88}{2{,}7} = 6{,}6$, but $v = 6{,}6$ gives more correctly $\zeta_1 = 0{,}0211$, hence we shall have more correctly:
$v = \frac{17{,}88}{\sqrt{1{,}505 + 288 \cdot 0{,}0211}} = \frac{17{,}88}{\sqrt{7{,}582}} = 6{,}50$ feet, and the quantity of water
$Q = 0{,}7854 \cdot \left(\frac{2}{12}\right)^2 \cdot 6{,}50 = 0{,}137$ cubic feet $= 236{,}7$ cubic inches.

3. What width must be given to a conducting pipe, 100 feet in length, which at a head of water of 5 feet, delivers half a cubic foot of water per second? Here
$d = 0{,}4787 \sqrt[5]{(1{,}505\, d + 100\, \zeta_1) \cdot \frac{1}{5}\, (\frac{1}{2})^2} = 0{,}4787 \sqrt[5]{0{,}075\, d + 5\, \zeta_1}$. If we set out with $\zeta_1 = 0{,}02$, we obtain $d = 0{,}4787 \sqrt[5]{0{,}075\, d + 0{,}100}$, or approximately
$= 0{,}4787 \sqrt[5]{0{,}100} = 0{,}30$, therefore more correctly,
$d = 0{,}4787 \sqrt[5]{0{,}0225 + 0{,}100} = 0{,}4787 \sqrt[5]{0{,}1225} = 0{,}31456$ feet $= 3{,}77$ inches. To this width corresponds the transverse section $F = 0{,}7854 \cdot 0{,}31456^2 = 0{,}0777$ square feet, the velocity $v = \frac{Q}{F} = \frac{0{,}5}{0{,}0777} = 6{,}43$ feet, and to this again the co-efficient of resistance $\zeta_1 = 0{,}0211$. If we substitute the last more accurate value, we then obtain $d = 0{,}4787 \sqrt[5]{0{,}1280} = 0{,}3173$ feet.

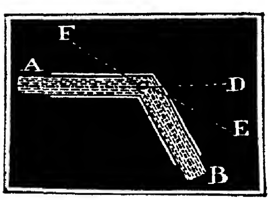
Remark. Experiments with $2\frac{1}{2}$ and $4\frac{1}{2}$ inch wide common wooden pipes have given the author the co-efficient of resistance 1,75 times as great as for metallic pipes, to which refer the values given in the table of the former §. Whilst, therefore, for example, for a velocity of 3 feet, $\zeta_1 = 0{,}0242$ for metallic pipes, we must put it for wooden pipes $= 0{,}0242 \cdot 1{,}75 = 0{,}04235$; whilst we have found in *Example* 1, the head of water for a metallic pipe 150 feet long, 1,527 feet, it will amount, under the same circumstances, for a wooden one to,

$$= (1{,}505 + 0{,}04235 \cdot 360) \cdot 0{,}0155 \cdot 9{,}339 = 16{,}75 \cdot 0{,}1494 = 244 \text{ feet.}$$

§ 332. *Bent Tubes.*—Particular resistances are opposed to the motion of water in pipes when they are *bent* or *knee-shaped.* Experiments have been made by the author on both kinds of resistances, on which account it is necessary here to communicate the results.

If a pipe ACB, Fig. 447, forms a knee, or if, as it is termed, it be *angled,* a loss of pressure ensues, which increases uniformly with the height $\frac{v^2}{2g}$ due to the velocity, and, further, increases with half the angle of deflexion $ACF = BCE = \delta$. The height of water lost through the knee, or the height due to the resistance corresponding to its transit through the knee, may be given by the expression $h = \zeta_2 \frac{v^2}{2g}$, where ζ_2 expresses the co-efficient of the knee resistance, dependent on the magnitude of the angle of deviation of the tube. Experiments made on different knees have led to the expression

Fig. 447.



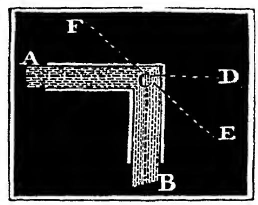
$$\zeta_2 = 0{,}9457\, sin.\, \delta^2 + 2{,}047\, sin.\, \delta^4,$$

and from this the following table has been calculated.

δ^0	10	20	30	40	45	50	55	60	65	70
ζ_2	0,046	0,139	0,364	0,740	0,984	1,260	1,556	1,861	2,158	2,431

It is seen from this, that a considerable loss of *vis viva* is produced by knee tubes; for example, a rectangular knee ACB, Fig. 448, gives, since the angle of deviation, amounts to 45°, the loss of head $= 0,956 \frac{v^2}{2g}$, therefore, pretty nearly equal to the height due to the velocity; a knee of 125°, for which $\delta = 62\frac{1}{2}°$, diminishes the head of water by so much as double the height due to the velocity $2 . \frac{v^2}{2g}$. By putting in the height due to the resistance of the knee $\zeta^2 \frac{v^2}{2g}$, we obtain the complete formula for the motion of water in tubes:

Fig. 448.



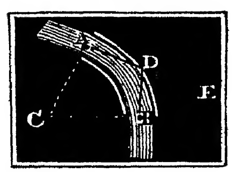
$$h = (1 + \zeta_1 \frac{l}{d} + \zeta_2) \frac{v^2}{2g}.$$

Example. If the conducting pipe in the first example of the preceding paragraph, 150 feet long and 5 inches wide, which is to deliver 25 cubic feet of water per minute, contains two rectangular knees (Fig. 451), we then have the required head of water:

$$h = (1,505 + 8,712 + 2 . 0,956) \frac{v^2}{2g} = 12,129 . 0,0155 . 9,339 = 12,129 . 0,14475$$
$$= 1,7557 \text{ feet.}$$

§ 334. *Curved Tubes.*—Curved tubes AB, Fig. 449, offer, under otherwise similar circumstances, far less resistance than unrounded knee tubes. The height due to the resistance which measures this obstacle increases likewise as the square of the velocity, but at the same time also as the simple angle of deflexion or curvature $ACB = BDE = \beta$, and may be expressed, therefore, by the formula:

Fig. 449.



$$h = \zeta . \frac{\beta^0}{180^0} . \frac{v^2}{2g} = \zeta . \frac{\beta}{\pi} . \frac{v^2}{2g}.$$

The corresponding co-efficient of resistance is by no means constant, it depends much more on the ratio of the width of the tube to the radius of curvature of its axis, and is the less, the less is this ratio. An extensive series of experiments made by the author, and the well known experiments of Du Buat, have, by their combinations, led to the expression $\zeta = 0,131 + 1,847 \left(\frac{r}{R}\right)^{\frac{7}{2}}$, for tubes with circular transverse sections, and for tubes with quadrangular or rectangular

transverse sections $\zeta = 0{,}124 + 3{,}104 \left(\frac{r}{R}\right)^{\frac{7}{2}}$, where r represents half the width of the tube, and R the radius of curvature of the axis.
The two following tables have been calculated accordingly.

TABLE I.

CO-EFFICIENTS OF THE RESISTANCE OF CURVATURE IN TUBES WITH CIRCULAR TRANSVERSE SECTIONS.

$\frac{r}{R}$	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
ζ	0,131	0,138	0,158	0,206	0,294	0,440	0,661	0,977	1,408	1,978

TABLE II.

CO-EFFICIENTS OF THE RESISTANCE OF CURVATURE IN TUBES WITH RECTANGULAR TRANSVERSE SECTIONS.

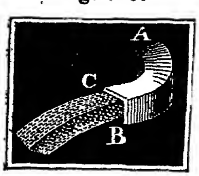
$\frac{r}{R}$	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
ζ	0,124	0,135	0,180	0,250	0,398	0,643	1,015	1,546	2,271	3,228

It is from hence seen, that for a round tube whose radius of curvature is twice as great as the radius of the tube, the co-efficient of resistance is $= 0{,}294$, and for a tube whose radius of curvature is at least ten times as great as the radius of the transverse section, this co-efficient $= 0{,}131$.

Example.—1. If the conducting pipe in the second example of § 332 has five small curves of 90° curvature, and of the ratio $\frac{r}{R} = \frac{1}{2}$ (Fig. 452), what quantity of water will it deliver? The height due to the resistance of the one curve
$= 0{,}294 \cdot \frac{90^\circ}{180^\circ} \frac{v^2}{2g} = 0{,}147 \cdot \frac{v^2}{2g}$; hence, for all five curvatures, it
$= 5 \cdot 0{,}147 \frac{v^2}{2g} = 0{,}735 \frac{v^2}{2g}$, and, accordingly, the velocity sought:
$v = \frac{17{,}88}{\sqrt{7{,}584 + 0{,}735}} = \frac{17{,}88}{\sqrt{8{,}319}} = 6{,}153$ feet, and the quantity of water:
$Q = 0{,}7854 \cdot \frac{1}{36} \cdot 6{,}130 = 0{,}1337$ cubic feet $= 231$ cubic inches.

2. If the curved buckets of a turbine form channels 12 inches long, 2 inches broad, and 2 inches deep, as ABC, Fig. 450, and if the water flows through them with a velocity of 50 feet, and the mean radius of curvature R of this axis of the channels amounts to 8 inches, then is $\frac{r}{R} = \frac{1}{8}$, the co-efficient of the resistance of curvature $= 0{,}127$; further, $\frac{\beta}{\pi} = \frac{12}{\pi r} = \frac{12}{8\,\pi} = 0{,}4774$; and lastly, the height due to the resistance corresponding to the curvature of the scoop

Fig. 450.



$$= 0{,}127 \,.\, 0{,}4774 \frac{v^2}{2g} = 0{,}0606 \frac{v^2}{2g} = 0{,}0606 \,.\, 0{,}0155 \,.\, 50^2 = 2{,}348 \text{ feet.}$$

Therefore, by the resistance of curvature, 2,348 feet in fall are lost.

Remark. The earlier formula given by Du Buat, Gerstner, and Navier for the resistance of curvature, are quite useless. An extended account of the experiments of the author on this subject will be published in the third number of his "Investigations in Mechanics and Hydraulics."

§ 335. *Jets d'Eau.*—A conducting tube either discharges into the air or under water. The discharge under water is applied when the tube at its outer orifice is so wide that the entrance of air may be feared. Here of course the head of water RC, Fig. 451, must be taken from the surface H of the upper water to that of C of the lower water. If the tube, for example, KLM, Fig. 452, discharges into the open air, it will give a stream of water OR, which, when allowed to ascend, is called a *jet d'eau.* We shall here consider what is most required for these jets. That a jet may ascend to the utmost possible height, it is necessary that the water should flow from the adjutage with great velocity; hence such adjutages must be applied which offer the fewest obstacles to the water in its passage, to which, therefore, the greatest co-efficients of velocity are due. Orifices in a thin plate, short tubes fashioned like the contracted fluid vein, and long and conically convergent ones, are those which give the greatest velocities of efflux. Orifices in a thin plate are little suitable, because a jet formed by them presents nodes and bulgings, and, therefore, is sooner scattered by the external air than the prismatic jet. The same takes place in a certain degree with short mouth-pieces, shaped like the contracted vein. Hence, for fountains and fire-engines, mostly long and slightly conically convergent, similar to those which d'Aubuisson used for his experiments, are very properly made use of. Sometimes entirely cylindrical jets are used. Where these mouth-pieces, as, for example, KL, Fig. 453, are screwed on to the conducting tube AB, they should gradually widen, that no contraction may occur in passing into them. If these mouth-pieces or discharging tubes are very long, like those of fire-engines, the friction of the water in them will then cause a considerable loss of pressure, because the water has here

Fig. 451.

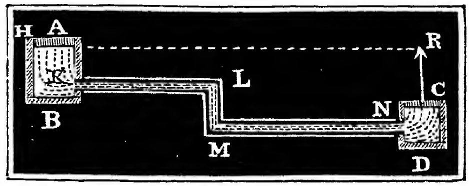


Fig. 452.

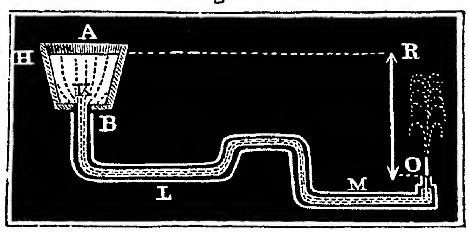
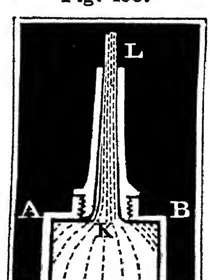


Fig. 453.



a great velocity. For great velocities we may properly put the co-efficient of resistance $\zeta = 0{,}016$, and, therefore, the loss of head of water $= 0{,}016\ \frac{l}{d} \cdot \frac{v^2}{2g}$. If now the length of a hose is twenty times as great as the mean width, we shall then obtain the height of the resistance due to friction

$$= 0{,}016 \,.\, 20 \,.\, \frac{v^2}{2g} = 0{,}32 \,.\, \frac{v^2}{2g};$$

thus, from this cause, above 32 per cent. of the height of ascent is lost. These tubes are generally much longer, hence this loss is greater.

The velocity with which water passes out of a mouth-piece or hose, and on which the jet or the height of ascent principally depends, may be estimated by means of the above principles. If we put this velocity of efflux $= v$, the width of the mouth-piece at the exit orifice $= d$, and the mean width of the conducting tube $= d_1$, we shall then obtain the velocity of the water in it $v_1 = \frac{d^2}{d_1^2} v$. If ζ represent the co-efficient of resistance at the inner orifice of the tube, ζ_1 that of the resistance of friction in the pipe, and ζ_2 the co-efficient for the knees or curvature of the pipe, the height due to the resistance for the motion of water in the conduit pipes will be:

$$h = (\zeta + \zeta_1 \frac{l}{d} + \zeta_2) \frac{v_1^2}{2g} = (\zeta + \zeta_1 \frac{l}{d} + \zeta_2) \frac{d^4}{d_1^4} \cdot \frac{v^2}{2g}.$$

It is seen from this, that the resistance to the water is less, the wider the conduit pipe is. It is hence an important rule, to employ as wide pipes and hoses as possible, for leading water to *jets d'eau* and for fire-engines.

If, further, we represent the co-efficient of resistance for the mouth-piece by ζ_3, we have then the height due to the resistance for this $= \zeta_3 \frac{v^2}{2g}$, and the sum of all the heights due to the resistance is then:

$$= \left[\left(\zeta + \zeta_1 \frac{l}{d} + \zeta_2\right) \frac{d^4}{d_1^4} + \zeta_3\right] \frac{v^2}{2g}.$$

If, lastly, the height of pressure, *i. e.* the depth RO, Fig. 452, of the outer orifice O below the surface of water H in the reservoir $= h$, the formula

$$h = \left[1 + \zeta_3 + (\zeta + \zeta_1 \frac{l}{d} + \zeta_2) \frac{d^4}{d_1^4}\right] \frac{v^2}{2g},$$

holds true, and hence the velocity of efflux is:

$$v = \frac{\sqrt{2gh}}{\sqrt{1 + (\zeta + \zeta_1 \frac{l}{d} + \zeta_2) \frac{d^4}{d_1^4} + \zeta_3}}.$$

If the jet were to rise perpendicularly and in vacuo, the height of ascent would be:

$$s = \frac{v^2}{2g} = \frac{h}{1 + (\zeta + \zeta_1 \frac{l}{d} + \zeta_2)\frac{d^4}{d_1^4} + \zeta_3},$$

but because the air and the descending water offer impediments to the ascent, and to the direction of the jet, as is the case in fire-engines, the effective height of ascent is somewhat less. According to d'Aubuisson's conclusions from the experiments undertaken upon this subject by Mariotte and Bossut, the effective height of ascent is $s_1 = s - 0{,}01 \,.\, s^2 = s\,(1 - 0{,}01 \,.\, s)$ metres, or for English measure $= s\,(1 - 0{,}00305\, s)$ feet.

We see from this that in great ascents proportionately more height is lost than in small velocities. Thick jets ascend somewhat higher than thin ones. In order to diminish the resistance of the descending water, the jet must be directed a little inclined. As to the height and amplitude of oblique jets, see § 38.

Example. If the conduit pipe for a fountain be 250 feet long, and 2 inches diameter, if the co-efficient of resistance corresponding to the mouth-piece $= 0{,}32$, if the entrance orifice at the reservoir be sufficiently rounded, and the bends that occur have sufficient radii of curvature to allow of our neglecting the corresponding co-efficients of resistance, to what height will a jet $\frac{1}{2}$ inch thick, under a head of water of 30 feet, rise? If we take the co efficient ζ_1 of friction $= 0{,}025$, we shall then obtain the entire height due to the resistance:

$$h = (1 + 0{,}025 \,.\, \frac{250}{\frac{1}{6}} \,.\, \left(\frac{\frac{1}{2}}{4}\right)^4 + 0{,}32)\frac{v^2}{2g} = 1{,}47 \,.\, \frac{v^2}{2g};$$

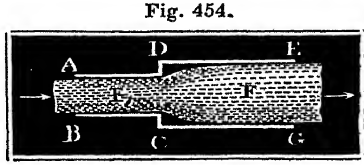
hence, the height due to the velocity $s = \frac{h}{1{,}47} = \frac{30}{1{,}47} = 20{,}41$ feet, and the effective height of ascent $s_1 = 20{,}41\,(1 - 0{,}00305 \,.\, 20{,}41) = 20{,}41 - 1{,}27 = 19{,}14$ feet.

CHAPTER IV.

ON THE RESISTANCES OF WATER IN PASSING THROUGH CONTRACTION.

§ 337. *Abrupt Widening.*—Changes in the transverse section of a tube, or of any other reservoir of efflux, produce changes in the velocity of the water. The velocity is inversely proportional to the transverse section of the stream. The wider the vessel is, the less is the velocity, and the narrower the vessel, the greater the velocity of the water flowing through it. If the transverse section of a vessel be suddenly altered, as, for example, in the tube *ACE*, Fig. 454, there then ensues a sudden alteration of the velocity, and this is accompanied by a loss of *vis viva*, or connected with a corresponding diminution of pressure. This loss may be as accurately measured as the mechanical effect in the impact of inelastic bodies (§ 258). Every particle of water which passes

Fig. 454.



from the narrower tube BD into the wider tube DG, strikes against the slowly moving mass of water in this tube, and, after impulse, joins itself to and proceeds onwards with it. It is exactly the same with the collision of solid and inelastic bodies; these bodies go on likewise after impact with a common velocity. Since we have already found that the loss of mechanical effect by the impact of these bodies is

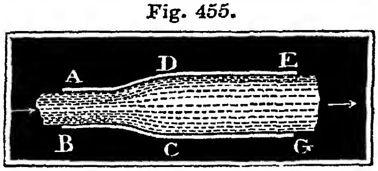
$$L = \frac{(v_1 - v_2)^2}{2g} \cdot \frac{G_1 G_2}{G_1 + G_2},$$

so we may here, as the impinging particle of water G_1 is indefinitely small compared with the impinged mass of water G_2, put: $L = \frac{(v_1 - v_2)^2}{2g} G_1$, and, consequently, the corresponding loss of head $h = \frac{(v_1 - v_2)^2}{2g}$.

There arises, therefore, from a sudden change of velocity a loss of head, which is measured by the height due to the velocity corresponding to this change.

If now the transverse section of the one tube AC, $= F_1$, and that of the other CE, $= F$, the velocity of the water in the first tube $= v_1$, and that in the other $= v$, we then have $v_1 = \frac{Fv}{F_1}$, hence the loss in head of water in the passage from one tube to the other is $h_1 = \left(\frac{F}{F_1} - 1\right)^2 \cdot \frac{v^2}{2g}$, and the corresponding co-efficient of resistance $\zeta = \left(\frac{F}{F_1} - 1\right)^2$.

Fig. 455.



The experiments undertaken by the author on this subject accord well with theory. That the tube DG may be filled with water, it is requisite that it be not very short, nor much wider than the tube AC. This loss vanishes, when, as represented in Fig. 455, a gradual passing from one tube into the other is accomplished by the rounding of the edges.

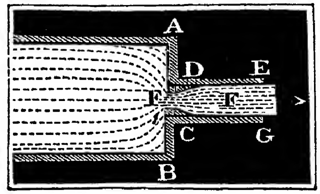
Remark. The head of water found $h_1 = \left(\frac{F}{F_1} - 1\right)^2 \frac{v^2}{2g}$ cannot, of course, be utterly lost, we must rather assume that the mechanical effect produced by it is expended on the separation of the previous continuity of the particles of water.

Example. If the diameter of a tube, of the construction in Fig. 454, is as great again as that of another tube, then is $\frac{F}{F_1} = \left(\frac{2}{1}\right)^2 = 4$, hence the co-efficient of resistance $\zeta = (4-1)^2 = 9$, and the corresponding height due to the resistance on passing from the narrow into the wide tube $= 9 \cdot \frac{v^2}{2g}$. If the velocity of the water in the latter tube $=$ 10 feet, the height due to the resistance is then $= 9 \cdot 0{,}0155 \cdot 10^2 = 13{,}95$ feet.

§ 338. *Abrupt Contraction.*—A sudden change of velocity also

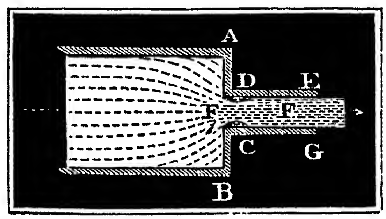
occurs when water passes from a cistern AB, Fig. 456, into a narrow tube DG, especially when there is a diaphragm at the place of entrance CD whose orifice is less than the transverse section of the tube DG. If the area of the contraction is $= F_1$, and α the co-efficient of contraction, we have then the transverse section F_2 of the contracted fluid vein $= \alpha F_1$; and if, on the other hand, F is the transverse section of the tube and v the velocity of efflux, we then find that the velocity at the contracted section F_2 is, $v_2 = \frac{F}{\alpha F_1} v$, and hence the loss of head in passing from F_2 into F, or from v_2 into v: $h = \frac{(v—v)^2}{2g} = \left(\frac{F}{\alpha F_1} - 1\right)^2 \frac{v^2}{2g}$, and the corresponding co-efficient due to the resistance: $\zeta = \left(\frac{F}{\alpha F_1} - 1\right)^2$.

Fig. 456.



Without the diaphragm, we have a mere short tube, Fig. 457; hence, $F = F_1$ and $\zeta = \left(\frac{1}{\alpha} - 1\right)^2$.

Fig. 457.



If we assume $\alpha = 0,64$, we then obtain:

$$\zeta = \left(\frac{1 - 0,64}{0,64}\right)^2 = (\tfrac{9}{16})^2 = 0,316.$$

But the co-efficient due to the resistance for the transit through an orifice in a thin plate is about 0,07; hence, here, where the water flows out $\frac{1}{\alpha}$ times as fast as from the contracted transverse section, the corresponding height due to the resistance $= 0,07 \cdot \left(\frac{v}{\alpha}\right)^2 \cdot \frac{1}{2g} = 0,07 \cdot \frac{1}{\alpha^2} \cdot \frac{v^2}{2g} \frac{0,07}{0,41} \cdot \frac{v^2}{2g} = 0,171 \cdot \frac{v^2}{2g}$.

By combining these two resistances, we obtain the entire height due to the resistance for efflux through a short tube:

$$= 0,316 \frac{v^2}{2g} + 0,171 \frac{v^2}{2g} = 0,49 \cdot \frac{v^2}{2g},$$

whilst we before found it $= 0,50 \frac{v^2}{2g}$.

Experiments on the efflux of water through an additional tube, with a narrow inner orifice, as in Fig. 456, have led the author to the following results. The co-efficient of resistance for transit through a diaphragm, and for a contraction at the wider tube, may be expressed by the formula $\zeta = \left(\frac{F}{\alpha F_1} - 1\right)^2$, but there must be put:

for $\frac{F_1}{F}$	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
α	0,616	0,614	0,612	0,610	0,607	0,605	0,603	0,601	0,598	0,596

And it follows that:

ζ	231,7	50,99	19,78	9,612	5,256	3,077	1,876	1,169	0,734	0,480

From this, for example, the co-efficient of resistance in the case where the narrow transverse section is half as great as that of the tube, is $\zeta = 5,256$, *i. e.*, for transit through this contraction, a head of water is required which is $5\frac{1}{4}$ times as great as the height due to the velocity.

Example. What discharge will the apparatus delineated in Fig. 456 give, if the head of water is $1\frac{1}{2}$ feet, the width of the circular contraction $1\frac{1}{2}$, and that of tube 2 inches? We have here $\frac{F_1}{F} = \left(\frac{1\frac{1}{2}}{2}\right)^2 = (\frac{3}{4})^2 = \frac{9}{16} = 0,56$, hence $\alpha = 0,606$, and $\zeta = \left(\frac{16}{9 \, . \, 0,606} - 1\right)^2 = \left(\frac{16 - 5,454}{5,454}\right)^2 = \left(\frac{10,546}{5,454}\right)^2 = 3,74$. If now we put $h = (1 + \zeta)\frac{v^2}{2g}$, we shall then obtain the velocity of efflux:

$v = \frac{\sqrt{2gh}}{\sqrt{1+\zeta}} = \frac{8,02\sqrt{1,5}}{\sqrt{4,74}} = 4,56$ feet, and consequently the quantity discharged:

$Q = \frac{\pi d^2}{4} v = \frac{\pi}{4} . 4 . 12 . 4,56 = 54,72 . \pi = 172$ cubic inches.

§ 339. *Effect of Imperfect Contraction.*—In the case considered in the last paragraph, the water issues from a large cistern, hence the contraction may be regarded as perfect; but if the transverse section of the cistern, or of the stream of fluid arriving at the narrow part, is not great with respect to the transverse section F_1, Fig. 458, of this contracted part; the contraction is then imperfect, and hence also the corresponding co-efficient of resistance is less than in the case above investigated. If we retain the former denominations, we have then also here the height due to the resistance, or the head of water expended by the transit through F_1, $h = \left(\frac{F}{\alpha F_1} - 1\right)^2 \frac{v^2}{2g}$, only, for α, we must substitute variable numbers which are greater the greater the ratio $\frac{G}{F_1}$ between the transverse section of the contraction and that of the conducting tube AB. If the diaphragm CD lies in a uniform tube AG, Fig. 459, then the same condition holds, only here the co-efficient α depends on $\frac{F_1}{F}$.

Fig. 458.

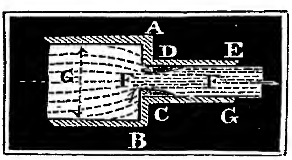
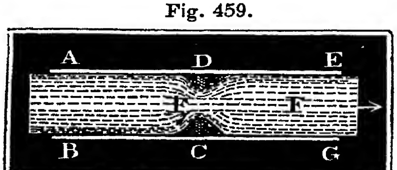


Fig. 459.



From experiments undertaken by the author, we must put into the formula $\zeta = \left(\frac{F}{\alpha F_1} - 1\right)^2$ for the co-efficient of resistance,

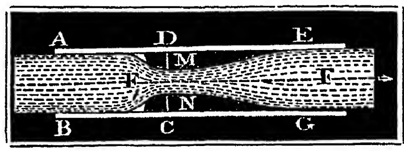
for $\frac{F_1}{F} =$	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
α	0,624	0,632	0,643	0,659	0,681	0,712	0,755	0,813	0,892	1,000

And it follows:

ζ	225,9	47,77	17,50	7,801	3,753	1,796	0,797	0,290	0,060	0,000

These losses are diminished when, by rounding off the edges, the contraction is diminished or counteracted, and they may be entirely neglected, if, as is represented in Fig. 460, a gradually widening tube MN is put on.

Fig. 460.



Example. What head of water is requisite that the apparatus in Fig. 461 may deliver 8 cubic feet of water per minute? If the width of the diaphragm $F_1 = 1\frac{1}{2}$, the width of the efflux tube $DG = 2$ inches, and the lower width of the afflux tube $AC = 3$ inches, we shall then have $\frac{F_1}{G} = \left(\frac{1\frac{1}{2}}{3}\right)^2 = \frac{1}{4}$, hence $\alpha = 0{,}637$; further, $\frac{F}{F_1} = \left(\frac{2}{1\frac{1}{2}}\right)^2 = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$, and the co-efficient of resistance:

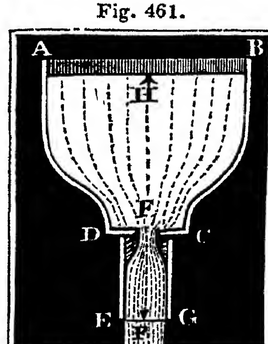
$= \left(\frac{16}{9\,.\,0{,}637} - 1\right)^2 = \left(\frac{10{,}267}{5{,}733}\right)^2 = 3{,}207$. The velocity of efflux is now:

$v = \frac{4\,Q}{\pi\, d^2} = \frac{4\,.\,8}{60\,.\,\pi\left(\frac{1}{6}\right)^2} = \frac{19.2}{\pi} = 6{,}112$ feet; hence, the head of water in question is

$h = (1 + \zeta)\,\frac{v^2}{2g} = 4{,}207\,.\,0{,}0155\,.\,6{,}112^2 = 2{,}43$ feet.

§ 340. *Slides, Cocks, Valves.*—For regulating the flow of water from pipes and cisterns, slides, cocks, valves, &c., are used, by which contractions are produced which offer obstacles to the passage of the water, and these may be determined in a manner similar to the loss estimated in the last paragraph. But since the water here undergoes further changes of direction, divisions, &c., the co-efficients α and ζ cannot be determined directly, but special experiments are necessary for this purpose. Such experiments have been also made,* and their principal results are communicated in the following tables.

Fig. 461.



* Experiments on the efflux of water through valves, slides, &c., undertaken and calculated by Julius Weisbach, under the title "Untersuchungen im Gebiete der Mechanik und Hydraulik," &c. Leipzig, 1842.

TABLE I.

THE CO-EFFICIENTS OF RESISTANCE TO THE PASSAGE OF WATER THROUGH SLIDING VALVES IN RECTANGULAR TUBES.

Ratios of transverse section $\frac{F_1}{F}$	1,0	0,9	0,8	0,7	0,6	0,5	0,4	0,3	0,2	0,1
Co-efficient of resistance ζ.	0,00	0,09	0,39	0,95	2,08	4,02	8,12	17,8	44,5	193

TABLE II.

THE CO-EFFICIENTS OF RESISTANCE TO THE PASSAGE OF WATER THROUGH SLIDES IN CYLINDRICAL TUBES.

Height of place s -	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$
Ratio of transverse section.	1,000	0,948	0,856	0,740	0,609	0,466	0,315	0,159
Co-efficient of resistance ζ.	0,00	0,07	0,26	0,81	2,06	5,52	17,0	97,8

TABLE III.

THE CO-EFFICIENTS OF RESISTANCE FOR THE PASSAGE OF WATER THROUGH A COCK IN A RECTANGULAR TUBE.

Angle of position.	5°	10°	15°	20°	25°	30°	35°	40°	45°	50°	55°	66¾
Ratio of transverse section.	0,926	0,849	0,769	0,687	0,604	0,520	0,436	0,352	0,269	0,188	0,110	0
Co-efficient of resistance.	0,05	0,31	0,88	1,84	3,45	6,15	11,2	20,7	41,0	95,3	275	∞

TABLE IV.

THE CO-EFFICIENTS OF RESISTANCE FOR THE PASSAGE OF WATER THROUGH A COCK IN A CYLINDRICAL TUBE.

Angle of position.	5°	10°	15°	20°	25°	30°	35°
Ratio of transverse section.	0,926	0,850	0,772	0,692	0,613	0,535	0,458
Co-efficient of resistance.	0,05	0,29	0,75	1,56	3,10	5,47	9,68

Angle of position.	40°	45°	50°	55°	60°	65°	$82\frac{1}{8}$°
Ratio of transverse section.	0,385	0,315	0,250	0,190	0,137	0,091	0
Co-efficient of resistance.	17,3	31,2	52,6	106	206	486	∞

TABLE V.

THE CO-EFFICIENTS OF RESISTANCE FOR THE PASSAGE OF WATER THROUGH THROTTLE VALVES IN RECTANGULAR TUBES.

Angle of position.	5°	10°	15°	20°	25°	30°	35°
Ratio of transverse section.	0,913	0,826	0,741	0,658	0,577	0,500	0,426
Co-efficient of resistance.	0,28	0,45	0,77	1,34	2,16	3,54	5,72

Angle of position.	40°	45°	50°	55°	60°	65°	70°	90°
Ratio of transverse section.	0,357	0,293	0,234	0,181	0,134	0,094	0,060	0
Co-efficient of resistance.	9,27	15,07	24,9	42,7	77,4	158	368	∞

TABLE VI.

THE CO-EFFICIENTS OF RESISTANCE FOR THE PASSAGE OF WATER THROUGH THROTTLE VALVES IN CYLINDRICAL TUBES.

Angle of position.	5°	10°	15°	20°	25°	30°	35°
Ratio of transverse section.	0,913	0,826	0,741	0,658	0,577	0,500	0,426
Co-efficient of resistance.	0,24	0,52	0,90	1,54	2,51	3,91	6,22

Angle of position.	40°	45°	50°	55°	60°	65°	70°	90°
Ratio of transverse section.	0,357	0,293	0,234	0,181	0,134	0,094	0,060	0
Co-efficient of resistance.	10,8	18,7	32,6	58,8	118	256	751	∞

§ 341. By means of the co-efficients derived from the above tables, we may not only assign the loss of pressure corresponding to a certain slide, cock, or position of a valve, but also deduce what position is to be given to this apparatus that the velocity of efflux or the resistance may be of a certain amount. Such a determination is, of course, the more to be relied on, the more the regulating arrangements are like to those used in the experiments. The numerical values given in the tables are only true for the case where the water, after its transit through the contractions produced by means of this apparatus, again fills this tube. That this full flow may take place for small contractions, the tube should have a considerable length. The transverse sections of the rectangular tubes were 5 centimetres (1,97 inch) broad and $2\frac{1}{2}$ (,98 inch) deep. The transverse sections of the cylindrical tubes had, however, a width of 4 centimetres (1,57 inch). By the slide, Fig. 462, there is a simple contraction, whose transverse section forms in the one tube a mere rectangular F_1, Fig. 463; in the second, however, a lune, F_1, Fig. 464. In the case of cocks, two

Fig. 462.

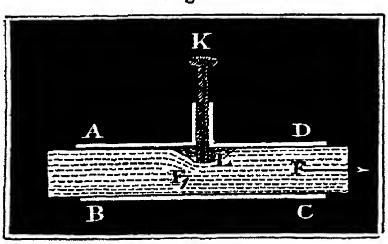


Fig. 463.

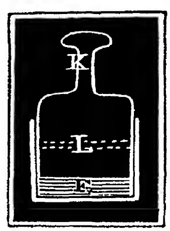
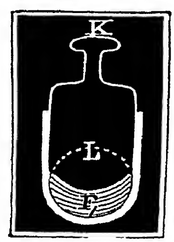


Fig. 464.



contractions present themselves, and also two changes of direction; on this account the resistances are also very considerable. The transverse sections of the greatest contractions have very peculiar figures. The stream in throttle valves, Fig. 466, divides itself into

Fig. 465.

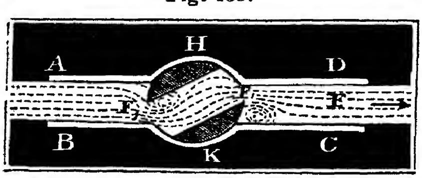
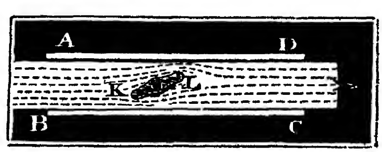


Fig. 466.



two portions, each of which passes through a contraction. The transverse sections of these contractions are, in the throttle valve of rectangular tubes, rectangular, and in cylindrical ones lunar-shaped. The following examples will suffice to show the application of the above tables.

Examples.—1. If a sliding valve is applied to a cylindrical conducting pipe 500 feet long and 3 inches wide, and this be drawn up to $\frac{3}{8}$ of its entire height, and therefore close $\frac{5}{8}$ of the pipe, what discharge will it deliver under a pressure of 4 feet? The co-efficient of resistance ζ for entrance into the pipe may be put from the above at 0,505, and that of the resistance of the slide from Table II. = 5,52, hence the velocity of the flow $v = \frac{8{,}02 \,.\, \sqrt{4}}{\sqrt{1{,}505 + 5{,}52 + \zeta_2 \frac{l}{d}}} = \frac{8{,}02 \,.\, 2}{\sqrt{7{,}025 + 500 \,.\, 4\,\zeta_2}} = \frac{16{,}04}{\sqrt{7{,}025 + 2000\,\zeta_2}}$. If we put the co-efficient of friction $\zeta_2 = 0{,}025$, we shall then obtain $v = \frac{16{,}04}{\sqrt{57{,}025}} = 2{,}12$ feet. But the velocity $v = 2{,}12$ feet, gives more correctly $\zeta_2 = 0{,}026$; hence, more accurately, $v = \frac{16{,}04}{\sqrt{59{,}025}} = 2{,}08$ feet, and the discharge per second $= \frac{\pi}{4} \,.\, 9 \,.\, 12 \,.\, 2{,}08 = 56{,}16 \,.\, \pi = 177$ cubic inches.—2. A conducting pipe, 4 inches wide, delivers, under a head of water of 5 feet, 10 cubic feet of water per minute; what position must be given to the throttle-valve applied, that it may afterwards deliver only 8 cubic feet? The velocity at the beginning is $= \frac{10 \,.\, 4}{60 \,.\, \pi \,(\frac{1}{3})^2} = \frac{6}{\pi} = 1{,}91$ feet, and after putting on the valve $\frac{8}{10} \,.\, 1{,}91 = 1{,}528$ feet. The co-efficient of efflux is $\frac{v}{\sqrt{2\,g\,h}} = \frac{1{,}91}{8{,}02\sqrt{5}} = 0{,}106$, hence the co-efficient of resistance $= \frac{1}{\mu^2} - 1 = \frac{1}{0{,}106^2} - 1 = 88$; the co-efficient of efflux for the second case is $= \frac{8}{10} \,.\, 0{,}106 = 0{,}0848$; hence the co-efficient of resistance $= \frac{1}{0{,}0848^2} - 1 = 138{,}0$, and consequently to produce the co-efficient of resistance of the throttle-valve: $\zeta = 138 - 88 = 50$. But now, from Table VI., the angle of position $\alpha = 50^\circ$, $\zeta = 32{,}6$, and the angle of position $\alpha = 55^\circ$, $\zeta = 58{,}8$; hence we may be allowed to assume, for a position of $50^\circ + \frac{15{,}7}{26{,}2} \,.\, 5^\circ = 53^\circ$, the desired quantity of discharge may be obtained. If we consider, further, for a change of velocity of 1,91 feet to 1,528 feet, the co-efficient of resistance passes from 0,0266 into 0,0281; then, more correctly, $\zeta = 138{,}0 - 88 \,.\, \frac{281}{266} = 138{,}06 - 92{,}96 = 45{,}04$, and, accordingly, the angle of position $= 50^\circ + \frac{10{,}9}{26{,}2} \,.\, 5^\circ = 52^\circ$.

§ 342. *Valves.*—The knowledge of the resistance produced by valves is of great importance. Experiments have been made by the author on this subject. The *conical* and *clack*, or *flap-valves*, Figs.

Fig. 467.

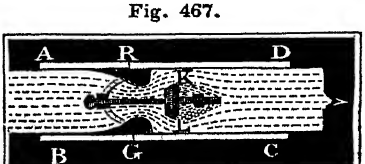
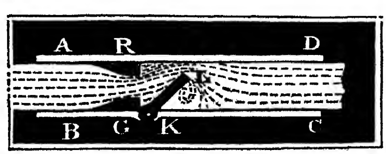


Fig. 468.



467 and 468, are those which most frequently are met with in practice. In both, the water passes through the aperture formed by a

ring RG; the conical valve KL has a guide rod, by which it is fixed in guides, and admits of an outward push only in the direction of the axis; the clack valve KL opens by turning like a door. It is easily seen in both apparatuses that a resistance is opposed to the water, not only by the valve ring, but also by the valve plate.

For the conical valve with which the experiments were undertaken, the ratio of the aperture in the valve ring, to the transverse section of the whole tube was 0,356, and, on the other hand, the ratio of the surface of the ring for the opened valve to the transverse section of the tube 0,406; hence, for the mean, we may put $\frac{F_1}{F} = 0{,}381$. Whilst the efflux in different positions of the valve was observed, it was found that the co-efficient of resistance diminished when the valve slide was greater, and that this diminution was almost insignificant when it exceeded half the width of the aperture. Its amount was in this case $= 11$, therefore, the height due to the resistance or the loss of head of water $= 11 \cdot \frac{v^2}{2g}$, v being the velocity of the water in the full tube. This number may be also used for determining the co-efficients of resistance corresponding to the other ratios of the transverse sections. Let generally $\zeta = \left(\frac{F}{\alpha F_1} - 1\right)^2$, we then obtain for the observed case $\frac{F_1}{F} = 0{,}381$, $\zeta = 11$, and $11 = \left(\frac{1}{0{,}381\,\alpha} - 1\right)^2$, hence

$$\alpha = \frac{1}{0{,}381\,(1 + \sqrt{11})} = \frac{1}{4{,}317 \cdot 0{,}381} = 0{,}608,$$

and, lastly, in general:

$$\zeta = \left(\frac{F}{0{,}608\,F_1} - 1\right)^2 = \left(1{,}645 \cdot \frac{F}{F_1} - 1\right)^2.$$

If, for example, the transverse section of the aperture is one half that of the tube, the co-efficient of resistance will accordingly $= (1{,}645 \cdot 2 - 1)^2 = 2{,}29^2 = 5{,}24$.

For the clack, or trap valve, the ratio of the transverse section of the aperture to the tube was $\frac{F_1}{F}, = 0{,}535$; but the following table shows in what degree the co-efficient of resistance diminishes with the size of the aperture.

TABLE

OF THE CO-EFFICIENTS OF RESISTANCE FOR TRAP VALVES.

Angle of aperture.	15°	20°	25°	30°	35°	40°	45°	50°	55°	60°	65°	70°
Co-efficient of resistance.	90	62	42	30	20	14	9,5	6,6	4,6	3,2	2,3	1,7

The co-efficients of resistance of these valves may be calculated approximately with the help of this table, even when the ratio of the transverse section is any other. The same method must be adopted as that followed for conical valves.

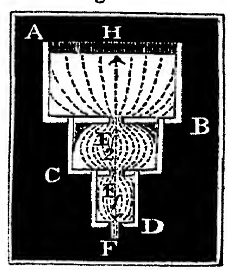
Example. A forcing-pump delivers by each descent of the piston, 5 cubic feet of water in 4 seconds, the width of the tube of ascent, in which lies the valve opening upwards, is 6 inches, the aperture of the valve-ring $3\frac{1}{2}$ inches, and the greatest diameter of the valve $4\frac{1}{2}$ inches; what resistance has the water in its passage through the valve to overcome? The ratio of the transverse section for the aperture is $\left(\frac{3,5}{6}\right)^2 = (\frac{7}{12})^2 = 0,34$, and that of the annular contraction to the transverse section of the tube $= 1 - \left(\frac{4,5}{6}\right)^2 = 1 - (\frac{3}{4})^2 = 0,44$, hence the mean ratio of section $= \frac{0,34 + 0,44}{2} = 0,39$ and the corresponding co-efficient of resistance $= \left(\frac{1,645\,.}{0,39} - 1\right)^2 = 3,22^2 = 10,4$. The velocity of the water is:

$$v = \frac{5}{4\,.\,\frac{\pi}{4}\,.\,(\frac{1}{2})^2} = \frac{20}{\pi} = 6,37$$

feet, the height due to the velocity $= 0,629$ feet; and, consequently, the resistance due to the height $= 10,4\,.\,0,629 = 6,54$ feet. The quantity forced up in one second weighs $\frac{5}{4}\,.\,62,5 = 77,6$ lbs.; hence the mechanical effect which by the transit of the water through the valve is consumed in this time $= 77,6 \cdot 6.54 = 507,5$ ft. lbs.

§ 343. *Compound Vessels.*—The principles already laid down on the resistance of water in its passage through contractions, find their application in the efflux of water through compound vessels. The apparatus *AD* represented in Fig. 469, is divided by two partition walls containing the orifices F_1 and F_2, and on this account forms three vessels of communication. Were there no partition walls, and the edges, where one vessel passes into the other, rounded off, we should then have as for a single vessel the velocity of flow through F: $v = \frac{\sqrt{2gh}}{\sqrt{1+\zeta}}$, h representing the depth FH below the surface of water, and ζ the co-efficient of resistance for the passage through the orifice F. But since obstacles are to be overcome on passing through F_1 and F_2, we then have to put $v = \sqrt{\frac{2gh}{1+\zeta+\zeta_1+\zeta_2}}$, and to substitute for ζ_1 and ζ_2, the co-efficients of resistance corresponding to the contractions F_1 and F_2. If we represent the transverse sections of the vessels *CD*, *BC* and *AB*, by G, G_1 and G_2, we may further put (§ 338):

Fig. 469.



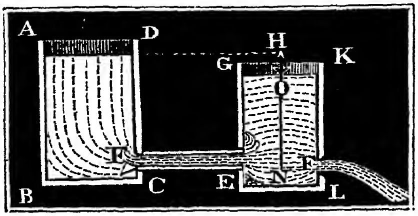
$$\zeta_1 = \left(\frac{G}{\alpha F_1} - 1\right)^2, \text{ and } \zeta_2 = \left(\frac{G_1}{\alpha F_2} - 1\right)^2,$$

hence also:

$$v = \frac{\sqrt{2gh}}{\sqrt{1+\zeta+\left(\frac{G}{\alpha F_1} - 1\right)^2 + \left(\frac{G_1}{\alpha F_2} - 1\right)^2}}.$$

Exactly the same relations take place in the compound apparatus of discharge represented in Fig. 470, except only that the friction of the water in the tube of communication *CE* has perhaps to be taken into account. If l is the length, and d the width of this tube, but ζ_1 the co-efficient of friction, and v_1 the velocity of the water in the tube of communication, we then have the height which the water loses in passing from *AC* to *GL*:

Fig. 470.



$$h_1 = \left[1 + \left(\frac{1}{\alpha} - 1\right)^2 + \zeta_1 \frac{l}{d}\right] \frac{v_1^2}{2g},$$

or, since the velocity is to be put:

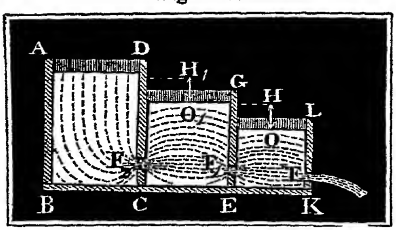
$$v_1 = \frac{\alpha F}{F_1} v,\ h_1 = \left[1 + \left(\frac{1}{\alpha} - 1\right)^2 + \zeta_1 \frac{l}{d}\right] \left(\frac{\alpha F}{F_1}\right)^2 \frac{v^2}{2g}.$$

If this height be deducted from the whole head of water h, there will remain the head of water in the second vessel $h_2 = h - h_1$, and hence the velocity of efflux:

$$v = \frac{\sqrt{2gh_2}}{\sqrt{1+\zeta}} = \frac{\sqrt{2gh}}{\sqrt{1 + \zeta + \left[1 + \left(\frac{1}{\alpha} - 1\right)^2 + \zeta_1 \frac{l}{d}\right]\left(\frac{\alpha F}{F_1}\right)^2}}.$$

This determination is very simple with the apparatus represented in Fig. 471, because the transverse sections G, G_1, G_2, of the cisterns may be made indefinitely great with respect to the transverse sections of the orifices F, F_1, F_2. Hence the first difference of level OH, or height due to the resistance in passing through:

Fig. 471.



$$F_1 \text{ is } h_1 = \frac{1}{2g}\left(\frac{v_1}{\alpha_1}\right)^2 = \left(\frac{\alpha F}{\alpha_1 F_1}\right)^2 \cdot \frac{v^2}{2g},$$

and likewise the second difference of level $O_1 H_1$, or the height due to the resistance in passing through

$$F_2 \text{ is } h_2 = \left(\frac{\alpha F}{\alpha_2 F_2}\right)^2 \cdot \frac{v^2}{2g},$$

where α, α_1, α_2, represent the co-efficients of contraction for the orifices F, F_1 and F_2. It accordingly follows that:

$$v = \frac{\sqrt{2gh}}{\sqrt{1 + \left(\frac{\alpha F}{\alpha_1 F_1}\right)^2 + \left(\frac{\alpha F}{\alpha_2 F_2}\right)^2}},$$

and the quantity of discharge:

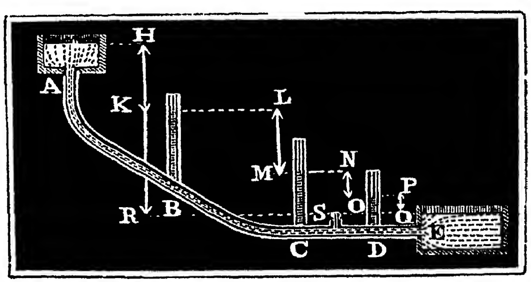
$$Q = \frac{\alpha F \sqrt{2gh}}{\sqrt{1 + \left(\frac{\alpha F}{\alpha_1 F_1}\right)^2 + \left(\frac{\alpha F}{\alpha_2 F_2}\right)^2}}$$
$$= \frac{\sqrt{2gh}}{\sqrt{\left(\frac{1}{\alpha F}\right)^2 + \left(\frac{1}{\alpha_1 F_1}\right)^2 + \left(\frac{1}{\alpha_2 F_2}\right)^2}}.$$

It is easy to perceive that compound reservoirs of efflux deliver less water, under otherwise similar circumstances, than simple ones.

Example. If in the apparatus, Fig. 470, the whole head of water or depth of the centre of the orifice F below the surface of water of the first cistern is 6 feet, the orifice 8 inches broad and 4 inches deep, the tube connecting both reservoirs 10 feet long, 12 inches broad, and 6 inches deep, what discharge will this reservoir give? The mean width of the tube $= \frac{4 \,.\, 1 \,.\, 0,5}{2 \,.\, 1,5} = \frac{2}{3}$ ft., hence $\frac{l}{d} = \frac{3 \,.\, 10}{2} = 15$; let us now put the co-efficient of friction $\zeta_1 = 0,025$, and it follows that $\zeta_1 \,.\, \frac{l}{d} = 0,025 \,.\, 15 = 0,375$; if the co-efficient of resistance for entrance into prismatic tubes be here put 0,505, we obtain $1 + \left(\frac{1}{\alpha_1} - 1\right)^2 + \zeta_1 \frac{l}{d} = 1 + 0,505 + 0,375 = 1,88$. As $\frac{\alpha F}{F_1} = \frac{0,64 \,.\, 8 \,.\, 4}{12 \,.\, 6} = 0,2845$, the co-efficient of resistance for the entire connecting tube $= 1,88 \,.\, 0,2845^2 = 0,152$, and the co-efficient of resistance for the transit through F, $= 0,07$, we then obtain the velocity of efflux: $v = \frac{8,02 \sqrt{6}}{\sqrt{1,07 + 0,152}} = \frac{8,02 \sqrt{6}}{\sqrt{1,222}} = 17,77$ feet. The contracted section is $0,64 \,.\, 1 \,.\, \frac{1}{2} = 0,32$ square feet, hence the discharge $= 0,32 \,.\, 17,77 = 5,68$ cubic feet.

§ 344. *Piëzometers.*—The loss of pressure which water suffers in conduit pipes from contractions, friction, &c., may be measured by columns of water, which are sustained in vertically placed tubes, which, when used for this purpose, are called *piëzometers.*

Fig. 472.



If v is the velocity of the water at a place B, Fig. 472, where a piëzometer is applied, l the length, d the width of the portion of tube AB, h the head of water or the depth of the point B below the surface of water; if, further, ζ is the co-efficient of resistance for entrance from the reservoir into the tube, and ζ_1 the co-efficient of friction, we then have for the height of the piëzometer measuring the pressure at B,

$$z = h - \left(1 + \zeta + \zeta_1 \frac{l}{d}\right) \frac{v^2}{2g}.$$

If the length of a portion of the tube $BC = l_1$, and its fall $= h_1$, we then have the height of the piëzometer at C:

$$z_1 = h + h_1 - \left(1 + \zeta + \zeta_1 \frac{l}{d} + \zeta_1 \frac{l_1}{d}\right) \frac{v^2}{2g},$$

and hence the difference of these two heights:

$$z_1 - z = h_1 - \zeta_1 \frac{l_1}{d} \cdot \frac{v^2}{2g},$$

hence, inversely, the height of the portion of tube BC, due to the resistance:

$$\zeta_1 \frac{l}{d} \cdot \frac{v^2}{2g} = h_1 + z - z_1 = \textit{the fall of the portion of tube}$$

plus the difference of the heights of the piëzometers.

From this it is seen that piëzometers are applicable to the measurement of the resistance which the water has to overcome in conduit pipes. If a particular impediment is found in the tubes; if, for instance, some small body is found fixed there, this will immediately be shown by the falling of the piëzometer, and the amount of the resistance produced, expressed. The resistances which are caused by regulating apparatus, such as cocks, slides, &c., may be likewise expressed by the height of the piëzometer. The piëzometer, for example, stands lower at D than at C, not only in consequence of the friction of the water in the portion of water CD, but also in consequence of the contraction which the slide S produces in this tube. If for a perfectly opened slide the difference NO of the height of the piëzometer $= h_1$, and for the slide partly closed $= h_2$, the new difference or depression $h_2 = h_1$ gives the height due to the resistance which corresponds to the passage of the water through the slide. Lastly, the velocity of efflux may be also estimated by the height of the piëzometer. If the height of the piëzometer $PQ = z$, the length of the last portion of tube $DE = l$, and its width $= d$, we then have:

$$z = \zeta_1 \frac{l}{d} \cdot \frac{v^2}{2g}, \text{ and hence } v = \sqrt{\frac{2gz}{\zeta_1 \frac{l}{d}}} = \sqrt{\frac{d}{l} \cdot \frac{2gz}{\zeta_1}}.$$

Example. If the height of the piëzometer $PQ = z$, Fig. 471, $\frac{3}{4}$ foot, and the length of the tube DE, measured from the piëzometer to the discharging orifice, $= 150$ feet, the width of the tube $3\frac{1}{2}$ inches, the velocity of efflux then follows:

$v = 8{,}02 \cdot \sqrt{\frac{3{,}5}{150 \cdot 12} \cdot \frac{0{,}75}{0{,}025}} = 8{,}02 \cdot 0{,}2415 = 1{,}937$ feet, and the discharge

$Q = \frac{\pi}{4} \cdot \left(\frac{3{,}5}{12}\right)^2 \cdot 1{,}937 = 01{,}274$ cubic feet.

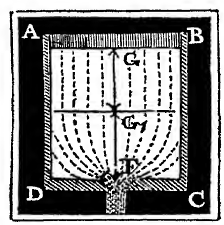
CHAPTER V.

ON THE EFFLUX OF WATER UNDER VARIABLE PRESSURE.

§ 345. *Prismatic Vessels.*—If a cistern from which water flows through an orifice at the side or bottom, has no influx to it from any other side, a gradual sinking of the surface of water will take place, and the cistern at last empty itself. If, further, the quantity of influx Q, be greater or less than the quantity of efflux $\mu F \sqrt{2gh}$, the surface of water will then rise or fall until the head of water $h = \frac{1}{2g}\left(\frac{Q}{\mu F}\right)^2$, and after this the head of water and the velocity of efflux will remain unaltered. Our problem, then, is to find how the time, the rise and fall of the water, and the emptying of vessels of given form and dimensions, depend on each other.

The efflux from a prismatic vessel presents the most simple case when it takes place through an opening in the bottom, and when there is no efflux from above or below. If x is the variable head of water FG_1, F the area of the orifice, and G the transverse section of the vessel AC, Fig. 473, we have then the theoretical velocity of efflux $v = \sqrt{2gx}$, the theoretical velocity of the falling surface of the water $= \frac{F}{G} v = \frac{F}{G}\sqrt{2gx}$, and the effective velocity $v_1 = \frac{\mu F}{G}\sqrt{2gx}$. At the commencement: $x = FG = h$, and at the end of the efflux $x = 0$, therefore, the initial velocity is:
$c = \frac{\mu F}{G}\sqrt{2gh}$, and the final velocity $c_1 = 0$.

Fig. 473.



It is seen from the formula $v_1 = \sqrt{2\left(\frac{\mu F}{G}\right)^2 g\, x}$, that the motion of the surface is uniformly retarded, and the measure of the retardation $p = \left(\frac{\mu F}{G}\right)^2 g$, hence we also know (§ 14), that this velocity $=0$, and the discharge ceases, when

$$t = \frac{c}{p} = \frac{\mu F}{G}\sqrt{2gh} \div \left(\frac{\mu F}{G}\right)^2 g = \frac{G}{\mu F}\sqrt{\frac{2gh}{g^2}}, \text{ i. e. } t = \frac{2G\sqrt{h}}{\mu F\sqrt{2g}}.$$

We may also put:

$$t = \frac{2\,G\,h}{\mu\,F\sqrt{2gh}} = \frac{2\,G\,h}{Q}.$$

and, according to this, assume that double the time is required for the

efflux of the discharge Gh through the orifice at the bottom F, under a head of water decreasing from h to 0, than under a uniform pressure.

As the co-efficient of efflux μ is not quite constant, but is greater for a diminution of pressure, we must, therefore, in calculations of this kind, substitute a mean value of this co-efficient.

Example. In what time will a rectangular cistern, of 14 square feet section, empty itself through a round orifice at the bottom, of 2 inches width, if the original head of water amount to 4 feet? The time of efflux would be theoretically:

$$t = \frac{2 \cdot 14 \sqrt{4}}{8{,}02 \cdot \frac{\pi}{4} (\frac{1}{6})^2} = \frac{2 \cdot 14 \cdot 144 \cdot 2}{8{,}02 \cdot \pi} = \frac{8064}{8{,}02 \cdot \pi} = 320'' = 5 \text{ min. } 20 \text{ sec.}$$

At the end of half the time of efflux, the head of water will be $= (\frac{1}{2})^2 \cdot h = \frac{1}{4} \cdot 4 = 1$ ft. Now the co-efficient of efflux, which corresponds to the head of water $= 1$ foot is 0,613, hence the effective time of discharge will be $= \frac{320''}{0{,}613} = 521'' = 8$ minutes, 41 seconds.

§ 346. *Vessels of Communication.*—Since for an initial head of water h_1, the time of efflux $t_1 = \frac{2 G \sqrt{h_1}}{\mu F \sqrt{2g}}$, and for an initial head of water h_2 this time $t_2 = \frac{2 G \sqrt{h_2}}{\mu F \cdot \sqrt{2g}}$, it then follows by subtraction, that the time within which the head of water passes from h_1 to h_2, and the surface of water sinks $h_1 - h_2$ is:

$$t = \frac{2 G}{\mu F \cdot \sqrt{2 g}} (\sqrt{h_1} - \sqrt{h_2}), \text{ or for the English foot measure:}$$

$$t = 0{,}249 \frac{G}{\mu F} (\sqrt{h_1} - \sqrt{h_2}).$$

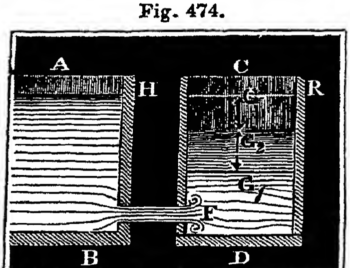
Inversely, the depression of the surface corresponding to a given time of efflux is $s = h_1 - h_2$, and is given by the formula:

$$h_2 = \left(\sqrt{h_1} - \frac{\mu \sqrt{2g} \cdot F}{2G} t\right)^2, \text{ or,}$$

$$s = \frac{\mu \sqrt{2g} \cdot F t}{G} \left(\sqrt{h_1} + \frac{\mu \sqrt{2g}}{4 G} \cdot t\right).$$

The same formulæ are further applicable, when a vessel CD, Fig. 474, is filled by another AB in which the water maintains a uniform height. If the transverse section of the tube of communication, or of the orifice $= F$, the transverse section of the vessel to be filled $= G$, and the original level $G G_1$ of the two surfaces of water $= h$, we have then, since here the surface of water G_1 in the second vessel is uniformly retarded, the time of filling likewise, or the time within which the second surface of water comes to the level HR of the first:

Fig. 474.



$$t = \frac{2\ G \sqrt{h}}{\mu\ F \cdot \sqrt{2 g}},$$

and likewise the time in which the height of level h_1 passes into h_2, and, therefore, the surface of water ascends to:

$$G G_1 = s = h_1 - h_2.$$

$$t = \frac{2\ G}{\mu\ F \cdot \sqrt{2 g}} (\sqrt{h_1} - \sqrt{h_2}).$$

Examples. 1.—How much will the surface of water in the vessel of the last example sink in two minutes? $h_1 = 4$, $t = 2 \cdot 60 = 120$, $\frac{F}{G} = \frac{\pi \cdot}{14 \cdot 144}$ and if we assume, further, $\mu = 0{,}605$, it follows then $h_2 = \left(\sqrt{h_1} - \mu \cdot \sqrt{2 g} \cdot \frac{F\ t}{2\ G}\right)^2$ $= \left(2 - \frac{0{,}605 \cdot 8{,}02 \cdot \pi \cdot 120}{2 \cdot 14 \cdot 144}\right)^2 = \left(2 - 0{,}605 \cdot 8{,}02 \cdot \frac{5 \cdot \pi}{168}\right)^2 = 2{,}393$ feet, and the depression sought is $s = 4 - 2{,}393 = 1{,}607$ feet.

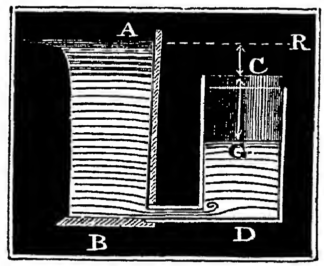
2. What time does the water in the 18 inch wide tube CD, Fig. 475, require to run over if it communicates with a vessel AB by a short $1\frac{1}{2}$ inch wide tube, and the rising surface of water G stands, at the beginning, 6 feet below the uniform surface of water A, and $4\frac{1}{2}$ feet below the head C of the tube. It is:

$$t = \frac{2\ G}{\mu \sqrt{2 g} \cdot F} (\sqrt{h_1} - \sqrt{h_2}),$$

$h_1 = 6$, $h_2 = 6 - 4{,}5 = 1{,}5$, $\frac{G}{F} = \left(\frac{18}{1{,}5}\right)^2 = 144$ and $\mu = 0.81$, whence it follows that:

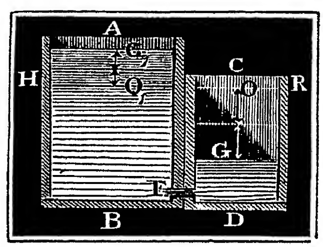
$$t = \frac{2 \cdot 144}{0{,}81 \cdot 8{,}02} (\sqrt{6} - \sqrt{1{,}5}) = \frac{288 \cdot 1{,}2248}{0{,}81 \cdot 8{,}02}$$
$= 54{,}3$ sec.

Fig. 475.



If the first vessel AB, Fig. 476, from which the water runs into the other, has no influx, and its section G_1 also not to be considered as indefinitely great compared with the section G of the subsequent vessel CD, we have then to modify the condition. If the variable distance $G_1 O_1$ of the first surface of water from the level HR at which both surfaces stand at the end of the efflux $= x$, and the distance GO of the second surface of water from this same plane $= y$, we have then the variable head of water $= x + y$, and the corresponding velocity of efflux: $v = \sqrt{2 g\ (x + y)}$, and the quantity of water:

Fig. 476.



$$G_1 x = G y,\ v = \sqrt{2 g \left(1 + \frac{G}{G_1}\right) y}.$$

The velocity with which the surface of water in the second vessel ascends is now:

$$v_1 = \frac{\mu\ F}{G} v = \frac{\mu\ F}{G} \sqrt{2 g \left(1 + \frac{G}{G_1}\right) y},$$

consequently the retardation:

$$p = \left(\frac{\mu F}{G}\right)^2 \left(1 + \frac{G}{G_1}\right) g,$$

and the time of efflux:

$$t = \frac{\mu F}{G} \sqrt{2 g \left(1 + \frac{G}{G_1}\right) y} \div \left(\frac{\mu F}{G}\right)^2 \left(1 + \frac{G}{G_1}\right) g$$

$$= \frac{2 G \sqrt{y}}{\mu F \sqrt{2 g \left(1 + \frac{G}{G_1}\right)}}.$$

Let us substitute for x and y, the initial height of level h, and therefore put:

$$x + y = h, \text{ or } \left(1 + \frac{G}{G_1}\right) y = h,$$

and we then obtain:

$y = \dfrac{h}{1 + \dfrac{G}{G_1}}$, and the time in which the two surfaces of water come to a level:

$$t = \frac{2 G \sqrt{h}}{\mu F \left(1 + \frac{G}{G_1}\right) \sqrt{2 g}} = \frac{2 G G_1 \sqrt{h}}{\mu F (G + G_1) \sqrt{2 g}}.$$

The time within which the level falls from h to h_1, is, on the other hand:

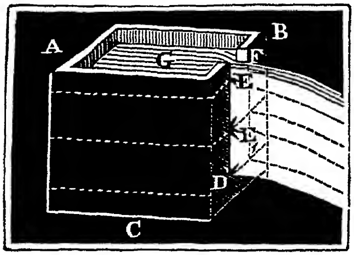
$$t = \frac{2 G G_1 (\sqrt{h} - \sqrt{h_1})}{\mu \sqrt{2 g} F (G + G_1)}.$$

Example. If the section of a cistern from which water flows is 10 square feet, and the section G of the recipient cistern 4 square feet; if, further, the initial level h of the two surfaces amounts to 3 feet, and the cylindrical tube of communication is 1 inch wide, then the time in which the water comes in both vessels to the same level is:

$$t = \frac{2 \,.\, 10 \,.\, 4 \,.\, \sqrt{3}}{0{,}82 \,.\, 8{,}02 \,.\, \frac{\pi}{4} \,.\, \frac{14}{144}} = \frac{320 \,.\, 72 \,.\, \sqrt{3}}{0{,}82 \,.\, 8{,}02 \,.\, 7\, \pi} = 275{,}9 \text{ sec.}$$

§ 348. *Notches in a Side.*—If water flows through the notch or cut DE of a prismatic cistern ABC, Fig. 477, to which there is no influx, the time of efflux may then be estimated in the following manner. Let us represent the transverse section of the cistern by G, the breadth EF of the notch by b, and the depth DE by h, and divide the whole aperture of efflux by horizontal lines into small slices, each of the breadth b and depth $\frac{h}{n}$. At a constant pressure the discharge per second will

Fig. 477.



be, $Q = \frac{2}{3} \mu\, b \sqrt{2 g h^3}$, if we divide this into the area $\frac{G h}{n}$ of a stratum of water, we shall then obtain the time of efflux $\tau = \frac{Gh}{\frac{2}{3} \mu\, nb \sqrt{2 g h^3}}$, which we may write:

$$\frac{3\, Gh}{2\, \mu\, n\, b \sqrt{2 g}} \cdot h^{-\frac{3}{2}}.$$

Now, to obtain the time of efflux t for a quantity of water $G (h - h_1)$, or to determine the time in which the head of water above the line $DE = h$ sinks to $DE_1 = h_1$, let us make $h_1 = \frac{m}{n} h$, and therefore h_1 to consist of m parts, and let us now substitute for $h^{-\frac{3}{2}}$, successively:

$$\left(\frac{m}{n} h\right)^{-\frac{3}{2}}, \left(\frac{m+1}{n} h\right)^{-\frac{3}{2}}, \left(\frac{m+2}{n} =\right)^{-\frac{3}{2}} \ldots \left(\frac{nh}{n}\right)^{-\frac{3}{2}},$$

and finally add the results obtained. In this manner we shall obtain the time required:

$$t = \frac{3\, G\, h}{2\, \mu\, n\, b \sqrt{2\, g}} \left[\left(\frac{mh}{n}\right)^{-\frac{3}{2}} + \left(\frac{m+1}{n}\, h\right)^{-\frac{3}{2}} + \ldots + \left(\frac{nh}{n}\right)^{-\frac{3}{2}}\right]$$

$$= \frac{3\, G\, h}{2\, \mu\, n\, b \sqrt{2\, g}} \cdot \frac{h^{-\frac{3}{2}}}{n^{-\frac{3}{2}}} \left(m^{-\frac{3}{2}} + (m+1)^{-\frac{3}{2}} + \ldots + n^{-\frac{3}{2}}\right)$$

$$= \frac{3\, Gh^{-\frac{1}{2}}}{2\, \mu\, n^{-\frac{1}{2}}\, b \sqrt{2g}} \left[\left(1^{-\frac{3}{2}} + 2^{-\frac{3}{2}} + 3^{-\frac{3}{2}} + \ldots + n^{-\frac{3}{2}}\right) - \left(1^{-\frac{3}{2}} + 2^{-\frac{3}{2}} + 3^{-\frac{3}{2}} + \ldots m^{-\frac{3}{2}}\right)\right],$$

or, from the "Ingenieur," Arithmetic, § 28:

$$t = \frac{3\, Gh^{-\frac{1}{2}}}{2\, \mu\, n^{-\frac{1}{2}}\, b \sqrt{2g}} \left(\frac{n^{-\frac{3}{2}-1}}{-\frac{3}{2}+1} - \frac{m^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right)$$

$$= \frac{3\, Gh^{\frac{1}{2}}}{2\, \mu\, b \sqrt{2\, gh}} \cdot 2 \left(m^{-\frac{1}{2}} - n^{-\frac{1}{2}}\right)$$

$$= \frac{3\, G}{\mu\, b \sqrt{2\, g\, h}} \left[\left(\frac{m}{n}\right)^{-\frac{1}{2}} - 1\right]$$

$$= \frac{3\, G}{\mu\, b \sqrt{2g}} \left[\left(\frac{m}{n}\, h\right)^{-\frac{1}{2}} - h^{-\frac{1}{2}}\right] = \frac{3\, G}{\mu b \sqrt{2g}} \left(\frac{1}{\sqrt{h_1}} - \frac{1}{\sqrt{h}}\right).$$

Let $h_1 = 0$, we have then $\frac{1}{\sqrt{h_1}}$, and therefore also $t = \infty$; an indefinite time, therefore, is required for the water to run down to the sill.

Example. If the water flows through a notch in a side, of 8 inches in breadth, from a reservoir 110 feet long and 40 feet broad, what time will it require to pass from a head of water of 15 inches to one of 6 inches?

$$t = \frac{3\,.\,110\,.\,40}{\mu\,.\,\frac{2}{3}\,.\,8{,}02}\left(\frac{1}{\sqrt{0{,}5}} - \frac{1}{\sqrt{1{,}25}}\right) = \frac{19800}{\mu\,.\,8{,}02}\left(\sqrt{2} - \sqrt{\tfrac{4}{5}}\right)$$

$$= \frac{19800}{8{,}02\,\mu}\,(1{,}4142 - 0{,}8944) = \frac{19800\,.\,0{,}5198}{8{,}02\,\mu} = \frac{1283}{\mu}\text{ sec.}$$

If we assume the co-efficient $\mu = 0{,}60$, the effective time of efflux will be $t = \frac{1283}{0{,}6} = 2138$ sec. $= 35$ min. 38 sec.

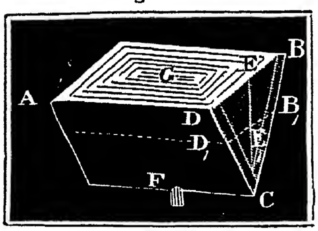
Remark. We may put for a rectangular lateral opening, approximatively:

$$t = \frac{2\,G}{\mu\,F\sqrt{2g}}\left[\left(\sqrt{h_1} - \sqrt{h_2}\right) - \frac{a^2}{288}\left(\sqrt{h_1^{-3}} - \sqrt{h_2^{-3}}\right)\right],$$

and F and G represent the transverse sections of the opening and of the vessel, a the depth of the opening, h_1 the head of water at the commencement, h_2 that at the end of the efflux. If $h_1 = \frac{a}{2}$, the opening becomes a notch, and we must then apply the proper formula.

§ 349. *Wedge and Pyramidal-shaped Vessels.*—If the cistern of discharge ABF, Fig. 478, forms a horizontal triangular prism, the time of efflux may be found in the following manner. Let us divide the height $CE = h$ into n equal parts, and carry horizontal planes through the points of division; let us then decompose the whole quantity of water into equally thick strata of equal length $AD = l$, and of breadths diminishing downwards. If the breadth of the upper stratum $BD = b$, we have then the breadth of another stratum D_1B_1, which stands about $CE_1 = x$ above the orifice F, lying at the lower edge, $b_1 = \frac{x}{h}\,b$, and its volume $= b_1\,l\,.\,\frac{h}{n} = \frac{b\,l\,x}{n}$. But now the discharge referred to a unit of time is: $Q = \mu\,F\sqrt{2\,g\,x}$, hence then the small time in which the surface of water sinks about $\frac{h}{n}$ is $\tau = \frac{bl}{n}x \div \mu\,F\sqrt{2\,g\,x} = \frac{b\,l}{n\,\mu\,F\sqrt{2g}}\,.\,\sqrt{x}$. Finally, since the sum of all the $x^{\frac{1}{2}}$ from $x = \frac{h}{n}$ to $x = \frac{nh}{n}$ are $= \left(\frac{h}{n}\right)^{\frac{1}{2}}\,.\,\frac{n^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}\,n\,h^{\frac{1}{2}}$, we have the time for the discharge of the entire prism of water:

Fig. 478.

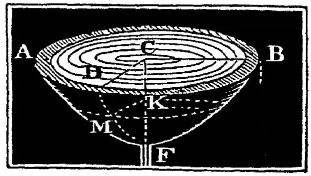


$$t = \frac{b\,l}{n\,\mu\,F\sqrt{2g}}\,.\,\tfrac{2}{3}\,n\,h^{\frac{1}{2}} = \tfrac{2}{3}\,\frac{b\,l}{\mu\,F\sqrt{2\,g}}\,.\,h^{\frac{1}{2}} = \tfrac{4}{3}\,\frac{\frac{1}{2}\,b\,l\,h}{\mu\,F\sqrt{2\,g\,h}}$$

$= \frac{4}{3}\,.\,\frac{V}{\mu\,F\,c}$, if V represents the whole quantity of water and c the initial velocity of efflux. Here the water, therefore, requires $\frac{1}{3}$ more time than if the velocity of efflux c were uniform.

If the vessel ABF, Fig. 479, forms an erect paraboloid, we then have for the ratio of the radii $KM = y$ and $CD = b$: $\frac{y}{b} = \frac{\sqrt{x}}{\sqrt{h}}$, and hence the ratio of the principal sections $\frac{G_1}{G} = \frac{y^2}{b^2} = \frac{x}{h}$, consequently $G_1 = \frac{Gx}{h}$ and the contents of a stratum of water $= G_1 \cdot \frac{h}{n} = \frac{Gx}{n}$. The perfect accordance of this expression with that found for the triangular prism, admits of our here putting $t = \frac{4}{3} \cdot \frac{\frac{1}{2}\, Gh}{\mu F \sqrt{2\, gh}}$, or, as $V = \frac{1}{2}\, G\, h$ (§ 118), also

Fig. 479.

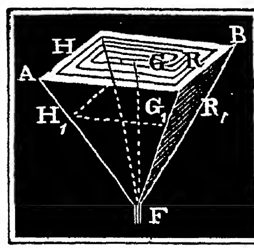


$$t = \frac{4}{3} \cdot \frac{V}{\mu\, F c}.$$

The formula may be used in many other cases for the approximative determination of the time of efflux, especially for that of the emptying of reservoirs. It is especially true in all cases where the horizontal sections increase as the distances from the bottom.

If, lastly, a vessel ABF be pyramidal, Fig. 480, then $G_1 : G = x^2 : h^2$, and hence $G_1 = \frac{Gx^2}{h^2}$ further the contents of the stratum $H_1\, R_1$: $\frac{G_1 h}{n} = \frac{Gx^2}{nh}$, and the time for its discharge:

Fig. 480.



$$\tau = \frac{Gx^2}{nh} : \mu\, F \sqrt{2\, gx} = \frac{G}{n \mu F h \sqrt{2\, g}} \cdot x^{\frac{3}{2}}.$$

But as the sum of all the $x^{\frac{3}{2}}$ taken from $x = \frac{h}{n}$ to $x = \frac{nh}{n} = \left(\frac{h}{n}\right)^{\frac{3}{2}} \cdot \frac{n^{\frac{5}{2}}}{\frac{5}{2}} = \frac{2}{5}\, n\, h^{\frac{3}{2}}$,

it follows that the time for the emptying of the whole pyramid is:

$$t = \frac{G}{n \mu F h \sqrt{2\, g}} \cdot \frac{2}{5}\, n\, h^{\frac{3}{2}} = \frac{2}{5} \cdot \frac{G h^{\frac{1}{2}}}{\mu F \sqrt{2\, g}} = \frac{6}{5} \cdot \frac{\frac{1}{3}\, Gh}{\mu F \sqrt{2\, gh}},$$

or if $\frac{1}{3}\, Gh$ be put $= V$, then will $t = \frac{6}{5} \cdot \frac{V}{\mu\, Fc}$.

As in this efflux the initial velocity of flow decreases gradually from c to zero, the time of efflux is then $\frac{1}{5}$th greater than if the velocity c remained uniform.

Example. In what time will a pond, whose surface has an area of 765000 square feet, empty itself, if there be a conduit 15 feet below the surface, and at the deepest place, which forms a channel 15 inches wide and 50 feet long? Theoretically, the time of

efflux is $t = \frac{4}{3} \cdot \frac{V}{F\sqrt{2gh}} = \frac{4}{3} \cdot \frac{1}{2} \cdot \frac{765000 \cdot 15}{\frac{\pi}{4} \cdot \left(\frac{5}{4}\right)^2 \cdot 8{,}02\sqrt{15}}$

$$= \frac{19584000}{\pi \cdot 8{,}02\sqrt{15}} = 200848 \text{ sec.}$$

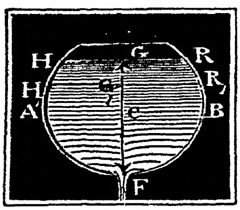
But now the co-efficient of resistance for entrance into the channel, inclined about 45°, is, $\zeta = 0{,}505 + 0{,}327$ (*see* § 323) $= 0{,}832$, and the resistance of the conduit due to friction $= 0{,}025\,\frac{l}{d} \cdot \frac{v^2}{2g} = 0{,}025 \cdot \frac{50}{\frac{5}{4}} \cdot \frac{v^2}{2g} = \frac{v^2}{2g}$; hence, the complete co-efficient of efflux for the channel is:

$\mu = \frac{1}{\sqrt{1 + 0{,}832 + 1}} = \frac{1}{\sqrt{2{,}832}} = 0{,}594$, and the time of efflux demanded:

$t = 200848 \div 0{,}594 = 338128 =$ 93 hours, 55 minutes, 28 seconds.

§ 350. *Spherical and Obelisk-shaped Vessels.*—By means of the formula of the last paragraph, we may now find the times of efflux for many other vessels, such as spherical, pontoon-shaped, pyramidal, &c. For the emptying of a spherical segment AB, Fig. 481, we obtain:

Fig. 481.



$$t = \frac{4}{3}\,\frac{\pi r h^2}{\mu F\sqrt{2gh}} - \frac{2}{5} \cdot \frac{\pi h^3}{\mu F\sqrt{2gh}}$$

$$= \frac{2}{15}\,\pi\,\frac{(10\,r - 3h)\,h^{\frac{3}{2}}}{\mu F\sqrt{2g}},$$

therefore, for the emptying of a full sphere, where $h = 2\,r$,

$$t = \frac{16\,\pi r^2\sqrt{2\,r}}{15\,\mu F\sqrt{2\,g}},$$

and for that of half a sphere where:

$$h = r\,t, = \frac{14\,\pi r^2\sqrt{r}}{15\,\mu F\sqrt{2g}}.$$

Here the horizontal stratum H_1R_1 corresponding to the depth $FG_1 = x = G_1 = \pi x\,(2\,r - x) \cdot \frac{h}{n} = \frac{2\,\pi r h x}{n} - \frac{\pi h x^2}{n}$, therefore:

$$\tau = \frac{2\,\pi r h}{n\,\mu F\sqrt{2\,g}} \cdot x^{\frac{1}{2}} - \frac{\pi h}{n\,\mu F\sqrt{2\,g}} \cdot x^{\frac{3}{2}};$$

as the first part of this expression agrees with the formula for the emptying of a prismatic, and the second part for the emptying of a pyramidal vessel, if we put first $2\,\pi r h$ in place of bl, and secondly πh^2 in place of G, we shall obtain by means of the difference of the times of emptying of a prismatic and pyramidal vessel, found in the former paragraph:

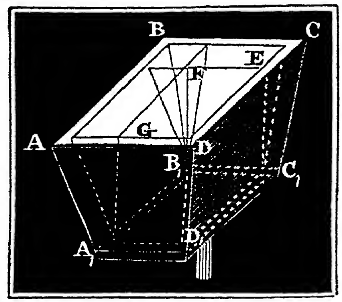
$$t = \frac{2}{3} \cdot \frac{b\,l\,h}{\mu F\sqrt{2gh}}, \text{ and } t = \frac{2}{5} \cdot \frac{G\,h}{\mu F\sqrt{2gh}},$$

the time also of the emptying of a spherical segment.

The above formula may be likewise applied to the case of an obelisk or pontoon-shaped vessel ACD_1, Fig. 482, since this is composed of a parallelopiped, two prisms, and a pyramid. Let b be the

breadth at top AD, and b_1 the breadth $A_1 D_1$ at bottom, l the length at top AB, and l_1 the length at bottom A_1B_1, and lastly, h the height of the vessel, we have then for the area of the surface $AC: bl = b_1 l_1 + b_1 (l - l_1) + l_1 (b - b_1) + (l - l_1)(b - b_1)$, of which $b_1 l_1$ belongs to the parallelopiped $A_1 C_1 E G$, $b_1 (l - l_1) + l_1 (b - b_1)$ to the two prisms CFB_1C_1, and AFB_1A_1 and $(l - l_1)(b - b_1)$ to the pyramid BFB_1. But now the time of efflux for the parallelopiped, whose base is $b_1 l_1$, is $t_1 = \frac{2\, b_1 l_1 \sqrt{h}}{\mu F \sqrt{2g}}$; further, that for the two triangular prisms

Fig. 482.



$$t_2 = \frac{2}{3} \frac{[b_1(l - l_1) + l_1 (b - b_1)] \sqrt{h}}{\mu F \sqrt{2g}}$$

and, lastly, for the pyramid:

$$t_3 = \frac{2}{5} \frac{(l - l_1)(b - b_1) \sqrt{h}}{\mu F \sqrt{2g}};$$

hence the time of discharge for the whole vessel is:

$$t = t_1 + t_2 + t_3$$

$$= [30\, b_1 l_1 + 10\, b_1 (l - l_1) + 10\, l_1 (b - b_1) + 6\, (l - l_1)(b - b_1)] \frac{\sqrt{h}}{15 \mu F \sqrt{2g}}$$

$$= [3\, bl + 8\, b_1 l_1 + 2\, (b l_1 + b_1 l)] \frac{2\sqrt{h}}{15 \mu F \sqrt{2g}}.$$

If $\frac{b_1}{l_1} = \frac{b}{l}$, we have then a truncated pyramid to consider. Let the one base $bl = G$, and the other $b_1 l_1 = G_1$, we then obtain:

$$t = (3\, G + 8\, G_1 + 4\sqrt{G G_1}) \frac{2\sqrt{h}}{15 \mu F \sqrt{2g}}.$$

It would be easy to show that this formula holds true also for every trilateral or multilateral pyramid.

Example. An obelisk-shaped water-cask is 5 feet long, and 3 feet broad at top, and at the depth of 4 feet, that is, at the level of a short horizontal discharge-tube, 1 inch in width, and 3 inches in length, it is 4 feet long and 2 feet broad, what time will be required for the water in the full cask to sink $2\frac{1}{2}$ feet? The time for emptying is, μ being taken $= 0{,}815$:

$$t = [8 \,.\, 4 \,.\, 2 + 3 \,.\, 5 \,.\, 3 + 2\, (3 \,.\, 4 + 5 \,.\, 2)] \frac{2\sqrt{4}}{15 \,.\, 0{,}815 \,.\, \frac{\pi}{4} \,.\, \left(\frac{1}{12}\right)^2 \,.\, 8{,}02}$$

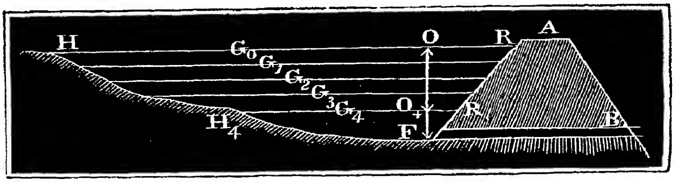
$= \frac{153 \,.\, 4 \,.\, 4 \,.\, 144}{15 \,.\, 0{,}815 \,.\, 8{,}02\, \pi} = 153 \,.\, \frac{2304}{12{,}225 \,.\, 8{,}02\, \pi} = 153 \,.\, 7{,}481 = 1145$ sec. As the level $4 - 2\frac{1}{2} = 1\frac{1}{2}$ feet above the tube $l = l_1 + \frac{3}{8} = 4\frac{3}{8}$ and $b = b_1 + \frac{3}{8} = 2\frac{3}{8}$ feet, hence the time for emptying if the vessel be filled only up to this level, is:

$$t_1 = [8 \,.\, 4 \,.\, 2 + 3 \,.\, \tfrac{35}{8} \,.\, \tfrac{19}{8} + 2\, (2 \,.\, \tfrac{35}{8} + 4 \,.\, \tfrac{19}{8})] \,.\, \frac{1152\sqrt{1{,}5}}{15 \,.\, 0{,}815 \,.\, 8{,}02\, \pi} = 131{,}672$$

$4,5779 = 602,76$ sec. The difference of the times found gives the time in which the surface of water originally at the top of the vessel sinks $2\frac{1}{2}$ feet.

§ 351. *Irregular Vessels.*—When we have to find the time of efflux for an irregularly formed vessel *HFR*, Fig. 483, we must apply Simp-

Fig. 483.



son's rule as a method of approximation. If we divide the whole mass of water into four equally thick strata, and the heads of water G_0, G_1, G_2, G_3, G_4, corresponding to the horizontal slices, represented by h_0, h_1, h_2, h_3, h_4, the time of efflux will be given by Simpson's rule

$$t = \frac{h_0 - h_4}{12\,\mu\, F \sqrt{2g}} \left(\frac{G_0}{\sqrt{h_0}} + \frac{4\,G_1}{\sqrt{h_1}} + \frac{2\,G_2}{\sqrt{h_2}} + \frac{4\,G_3}{\sqrt{h_3}} + \frac{G_4}{\sqrt{h_4}}\right).$$

In assuming six strata:

$$t = \frac{h_0 - h_6}{18\mu F\sqrt{2g}}\left(\frac{G_0}{\sqrt{h_0}} + \frac{4G_1}{\sqrt{h_1}} + \frac{2G_2}{\sqrt{h_2}} + \frac{4G_3}{\sqrt{h_3}} + \frac{2G_4}{\sqrt{h_4}} + \frac{4G_5}{\sqrt{h_5}} + \frac{G_6}{\sqrt{h_6}}\right).$$

The discharge in the first case is:

$$Q = \frac{h_0 - h_4}{12}\,(G_0 + 4\,G_1 + 2\,G_2 + 4\,G_3 + G_4),$$ in the second:

$$Q = \frac{h_0 - h_6}{18}\,(G_0 + 4\,G_1 + 2\,G_2 + 4\,G_3 + 2\,G_4 + 4\,G_5 + G_6).$$

When the form and dimensions of the vessel of efflux are not known, we may then calculate very nearly the discharge by the heads of water noted in equal intervals of time. Let t be one such interval, we have then for apertures at the bottom and sides:

$$Q = \frac{\mu F t \sqrt{2g}}{3}\,(\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}),$$

and for divisions or notches in a side:

$$Q = \tfrac{2}{3}\,\mu\, b\, t \sqrt{2g}\,(\sqrt{h^3_0} + 4\sqrt{h^3_1} + 2\sqrt{h^2_2} + 4\sqrt{h^3_3} + \sqrt{h^3_4}).$$

Example. In what time will the surface of water in a pond sink 6 feet, if the sluice forms a half cylinder, 18 inches wide, 9 inches deep, and 60 feet long, and the surfaces of water have the following areas?

G_0, at 20	feet head of water,	$= 600000$	square feet.
G_1, " 18,5	" "	$= 495000$	"
G_2, " 17,0	" "	$= 410000$	"
G_3, " 15,5	" "	$= 325000$	"
G_4, " 14,0	" "	$= 265000$	"

$F = \frac{\pi}{8}\cdot(\frac{3}{2})^2 = \frac{9\pi}{32} = 0,8836$ square feet. Let the co-efficient of resistance for the entrance $= 0,832$, and that for the friction:

$= 0,025\,.\,\frac{l}{d} = 0,025\,.\,60\,.\,1,091 = 1,6356$, then is the co-efficient of efflux

$$\mu = \frac{1}{\sqrt{1 + 0,832 + 1,6365}} = \frac{1}{\sqrt{3,4685}} = 0,537,$$

and $\mu F \sqrt{2g} = 0{,}537 \,.\, 0{,}8836 \,.\, 8{,}02 = 3{,}8054$. Now

$$\frac{G_0}{\sqrt{h_0}} = \frac{600000}{\sqrt{20}} = 134170,\ \frac{G_1}{\sqrt{h_1}} = \frac{495000}{\sqrt{18{,}5}} = 115090,$$

$$\frac{G_2}{\sqrt{h_2}} = \frac{410000}{\sqrt{17}} = 99440,\ \frac{G_3}{\sqrt{h_3}} = \frac{325000}{\sqrt{15{,}5}} = 82550,$$

$\frac{G_4}{\sqrt{h_4}} = \frac{265000}{\sqrt{14}} = 70830$; hence, then, the time of efflux follows:

$$t = \frac{6}{12 \,.\, 3{,}8054}(134170 + 4 \,.\, 115090 + 2 \,.\, 99440 + 4 \,.\, 82550 + 70830)$$

$$= \frac{1194440}{7{,}6108} = 156940 \text{ sec.}, = 43 \text{ hours, } 35 \text{ min. } 40 \text{ sec.}$$

The discharge is:

$$Q = \tfrac{6}{12}(600000 + 4 \,.\, 495000 + 2 \,.\, 410000 + 4 \,.\, 325000 + 265000)$$

$$= \frac{4965000}{2} = 2482500 \text{ cubic feet.}$$

§ 352. *Influx and Efflux.*—If the vessel during the efflux from below has an influx to it from above, the determination of the time in which the surface of water rises or falls a certain height becomes more complicated, so that we must be satisfied generally with but an approximate determination. If the discharge per second Q_1 is $> \mu F \sqrt{2gh}$, then there is a rise, and if $Q_1 < \mu F \sqrt{2gh}$, a fall of the surface. Moreover, a state of permanency occurs whenever the head of water is increased or decreased by $k = \frac{1}{2g}\left(\frac{Q_1}{\mu F}\right)^2$. The time τ, in which the variable head of water x increases by the small amount ξ, is given by the equation

$$G_1 \xi = Q_1 \tau - \mu F \sqrt{2gx} \,.\, \tau,$$

and, on the other hand, the time in which it sinks the height k, by

$$G_1 \xi = \mu F \sqrt{2gx} \,.\, \tau - Q_1 \tau.$$

Hence we have in the first case $\tau = \frac{G_1 \xi}{Q_1 - \mu F \sqrt{2gx}}$, and in the second $\tau = \frac{G_1 \xi}{\mu F \sqrt{2gx} - Q_1}$. By the application of Simpson's rule we then obtain the time of efflux, during which the lowering surface passes from G_0 to G_1, G_2 . . ., and the head of water from h_0 to h_1, h_2 . . .

$$t = \frac{h_0 - h_4}{12}\left[\frac{G_0}{\mu F\sqrt{2gh_0} - Q_1} + \frac{4\,G_1}{\mu F\sqrt{2gh_1} - Q_1} + \frac{2\,G_2}{\mu F\sqrt{2gh_2} - Q_1} + \frac{4\,G_3}{\mu F\sqrt{2gh_3} - Q_1} + \frac{2\,G_4}{\mu F\sqrt{2gh_4} - Q_1}\right],$$

or, more simply, if we represent $\frac{Q_1}{\mu F \sqrt{2g}}$ by $\sqrt{k}$,

$$= \frac{h_0 - h_4}{12 \mu F \sqrt{2g}}\left[\frac{G_0}{\sqrt{h_0} - \sqrt{k}} + \frac{4\,G_1}{\sqrt{h_1} - \sqrt{k}} + \frac{2\,G_2}{\sqrt{h_2} - \sqrt{k}} + \frac{4\,G_3}{\sqrt{h_3} - \sqrt{k}} + \frac{G_4}{\sqrt{h_4} - \sqrt{k}}\right].$$

If the vessel is prismatic, and has a uniform transverse section G, we then have:

$$t = \frac{2\,G}{\mu\,F\,\sqrt{2g}} \left(\sqrt{h} - \sqrt{h_1} + \sqrt{k}\ .\ hyp.\ log. \left(\frac{\sqrt{h} - \sqrt{k}}{\sqrt{h_1} - \sqrt{k}}\right)\right),$$

the time in which the head of water passes from h to h_1. Since for:

$$h_1 = k,\ \frac{\sqrt{h} - \sqrt{k}}{\sqrt{h_1} - \sqrt{h}} = \frac{\sqrt{h} - \sqrt{k}}{0} = \infty,$$

it follows that the condition of permanency takes place indefinitely late.

The following formula is the result of investigation for a wier or notch in a side.

$$t = \frac{G\,k}{3\,Q_1}\Big[hyp.\ log.\ \frac{(\sqrt{h} - \sqrt{k})^2\,(h_1 + \sqrt{h_1 k} + k)}{(\sqrt{h_1} - \sqrt{k})^2\,(h + \sqrt{hk} + k)}$$
$$+ \sqrt{12}\ .\ arc\ \left(tang. = \frac{(\sqrt{h} - \sqrt{h_1})\,\sqrt{12 k}}{3\,k + (2\,\sqrt{h} + \sqrt{k})\,(2\,\sqrt{h_1} + \sqrt{k})}\right)\Big],$$

where $k = \left(\frac{Q_1}{\frac{2}{3}\,\mu\,b\,\sqrt{2\,g}}\right)^{\frac{2}{3}}$, *hyp. log.* represents the hyperbolic logarithm, and *arc* ($tang. = y$) the arc whose tangent $= y$. According as k is $\lessgtr h$, and the inflowing quantity of water: $Q_1 \gtrless \frac{2}{3}\,\mu\,b\,\sqrt{2\,g\,h^3}$, there is a rise or fall of the fluid surface. The condition of permanency occurs, when $h_1 = k$, and the time corresponding becomes ∞.

Example. In what time will the water in a rectangular tank 12 feet long and 6 feet broad rise from 0 to 2 feet above the edge of a notch $\frac{1}{2}$ foot broad, if 5 cubic feet of water flow in per second? We have here $h = 0$; hence, more simply:

$$t = \frac{G\,k}{3\,Q_1}\left[hyp.\ log.\ \frac{h_1 + \sqrt{h_1\,k} + k}{(\sqrt{h_1} - \sqrt{k})^2} + \sqrt{12}\ arc\ \left(tang. = \frac{-\sqrt{3\,h_1}}{2\,\sqrt{k} + \sqrt{h_1}}\right)\right].$$

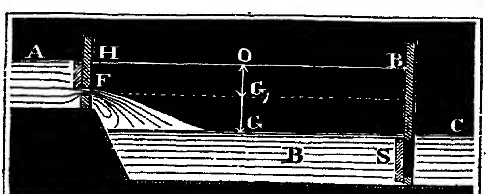
Now $G = 12\,.\,6 = 72$, $Q_1 = 5$, $h_1 = 2$, $b = \frac{1}{2}$, and $\mu = 0{,}6$,

$k = \left(\frac{5}{\frac{2}{3}\,.\,0{,}6\,.\,\frac{1}{2}\,.\,8{,}02}\right)^{\frac{2}{3}} = 2{,}1338$, and the time sought is:

$$t = \frac{72\,.\,2{,}1338}{3\,.\,5}\left[hyp.\ log.\ \frac{4{,}1338 + \sqrt{4{,}2676}}{(1{,}4142 - 1{,}4607)^2} - \sqrt{12}\,.\,arc\ \left(tang. = \frac{\sqrt{6}}{1{,}4142 + 2{,}9214}\right)\right]$$
$$= 10{,}242\left[hyp.\ log.\ \frac{6{,}1996}{0{,}002162} - \sqrt{12}\,.\,arc\ \left(tang. = \frac{\sqrt{6}}{4{,}3356}\right)\right]$$
$$= 10{,}242\ (7{,}961 - 1{,}781) = 10{,}242\,.\,6{,}18 = 63{,}29\ \text{sec.}$$

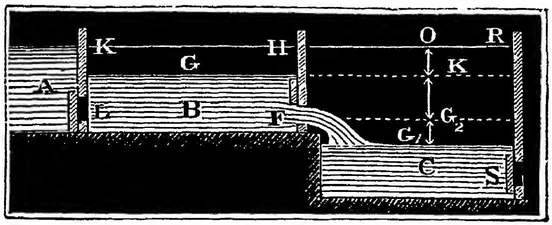
§ 353. *Locks.*—A very useful application of the doctrines hitherto treated of may be made to the filling and emptying of canal locks. We distinguish two kinds of navigation locks, single and double. The single lock, Fig. 484, consists of a chamber B, which is separated by the upper gate HF from the upper reach A, and by the

Fig. 484.



lower gate RS from the lower reach C. The double lock, Fig. 485, on the other hand, consists of two chambers, with the upper gate KL, the middle one HF, and the lower one RS.

Fig. 485.



Let the mean horizontal transverse section of a simple lock chamber $= G$, the distance of the middle of the sluice in the upper gate from the upper surface HR of the upper reach $= h_1$, and from that of the lower reach $= h_2$, and, lastly, the area of the aperture or sluice opening $= F$, we then obtain the time of filling up to the middle of the aperture $t_1 = \frac{G h_2}{\mu\, F \sqrt{2 g h_1}}$, and the time for filling the remaining space, where a gradual diminution of the head of water takes place, $t_2 = \frac{2\, G h_1}{\mu\, F \sqrt{2 g h_1}}$; consequently, the time for filling the single sluice is;

$$t = t_1 + t_2 = \frac{(h_2 + 2\, h_1)\, G}{\mu\, F \sqrt{2 g h_1}}.$$

If the aperture in the lower gate is entirely under water, then while emptying, the head of water gradually decreases from $h_1 + h_2$ to zero, hence the time for emptying or running off is:

$$t = \frac{2\, G \sqrt{h_1 + h_2}}{\mu\, F \sqrt{2 g}}.$$

If, on the other hand, a part of the aperture stands above the lower water, we then have two discharges to take into account; the one flowing above and the other below the water. Let the height of the part of the aperture above the water $= a_1$, and that under the water $= a_2$, the breadth of the aperture $= b$, we then obtain the time of efflux from the expression:

$$t = \frac{2\, G\, (h_1 + h_2)}{\mu\, b \sqrt{2g} \left(a_1 \sqrt{h_1 + h_2 - \frac{a_1}{2}} + a_2 \sqrt{h_1 + h_2} \right)}.$$

In double locks, the head of water gradually decreases in the chamber which is closed from the upper reach, during the discharge into the second chamber. If G is the horizontal transverse section of the first chamber, and the original head of water h_1 in this chamber sinks to x, whilst the water in the second chamber rises to the middle of the aperture of the sluice, we have then the corresponding time $t_1 = \frac{2\, G}{\mu\, F \sqrt{2 g}} (\sqrt{h_1} - \sqrt{x})$. Now the quantity of water

$G(h_1 - x) = G_1 h_2$, hence $x = h_1 - \frac{G_1}{G} h_2$, and

$$t_1 = \frac{2\,G}{\mu F \sqrt{2g}} \left(\sqrt{h_1} - \sqrt{h_1 - \frac{G_1 h_2}{G}} \right) = \frac{2\sqrt{G}}{\mu F \sqrt{2g}} \left(\sqrt{G h_1} - \sqrt{G h_1 - G_1 h_2} \right).$$

The time in which the water rises as high in the second as in the first chamber, and in which, therefore, it comes to the same level in both, may be found from § 347:

$$t_2 = \frac{2\,G G_1 \sqrt{x}}{\mu F (G + G_1) \sqrt{2g}} = \frac{2\,G_1 \sqrt{G} \sqrt{G h_1 - G_1 h_2}}{\mu F (G + G_1) \sqrt{2g}},$$

and the whole time for filling:

$$t = t_1 + t_2 = \frac{2\sqrt{G}}{\mu F \sqrt{2g}} \left(\sqrt{G h_1} - \frac{G}{G + G_1} \sqrt{G h_1 - G_1 h_2} \right).$$

Example. What time is required for the filling and running off of the following single lock chamber? The mean length of the lock $= 200$ feet, mean breadth $= 24$ feet, therefore $G = 200 \,.\, 24 = 4800$ square feet, distance of the centre of the aperture of the sluice in the upper gate from the two surfaces of water 5 feet, breadth of both apertures $2\frac{1}{2}$ feet, height of the aperture in the upper gate 4 feet, and of that in the lower gate (entirely under water) 5 feet. Let $t = \frac{(2\,h_1 + h_2)\,G}{\mu\,F\,\sqrt{2\,g\,h_1}}$, $h_1 = 5$, $h_2 = 5$, $G = 4800$, $\mu = 0{,}615$, $F = 4 \,.\, 2\frac{1}{2} = 10$, $\sqrt{2\,g} = 8{,}02$, we then obtain the time of filling:

$t = \frac{3 \,.\, 5 \,.\, 4800}{6{,}15 \,.\, 8{,}02 \sqrt{5}} = \frac{14400}{1{,}23 \,.\, 8{,}02 \sqrt{5}} = 652{,}85$ seconds. If we substitute in the formula $t = \frac{2\,G \sqrt{h_1 + h_2}}{\mu\,F\,\sqrt{2\,g}}$, $G = 4800$, $h_1 + h_2 = 10$, $F = 5 \,.\, 2\frac{1}{2} = 12{,}5$, we then obtain the time for emptying of the sluice:

$t = \frac{2 \,.\, 4800 \sqrt{10}}{0{,}615 \,.\, 12{,}5 \,.\, 8{,}02} = 491{,}78$ sec. $= 8$ min. $21{,}78$ sec.

CHAPTER VI.

ON THE EFFLUX OF AIR FROM VESSELS AND TUBES.

§ 354. *Efflux of Still Air.*—Condensed air does not flow from vessels quite in accordance with the law which regulates the flow of water, because an expansion takes place during its discharge, which is not the case with water. But in order to discover a similar law for air and other gases, let us make the mechanical effect $Q\gamma \frac{v^2}{2g}$, which a quantity of air Q of the density γ requires to pass from a state of rest into that of the velocity v, equal to the mechanical effect $Q\, p$ *hyp. log.* $\left(\frac{p_1}{p}\right)$ found in § 298, which the same quantity of air produces

when it passes from a greater pressure p_1 to a less p. If, therefore, p_1 be the elastic force of air enclosed in a vessel, v its velocity of efflux for the tension of the external air, and γ its density, then $Q\gamma \cdot \frac{v^2}{2g} = Qp\ hyp.\ log. \left(\frac{p_1}{p}\right)$, therefore, *the height due to the velocity:*

$$\frac{v^2}{2g} = \frac{p}{\gamma}\, hyp.\ log. \left(\frac{p_1}{p}\right) = 2{,}3026\, \frac{p}{\gamma}\, log. \left(\frac{p_1}{p}\right);$$

and the velocity itself:

$$v = \sqrt{2g\, \frac{p}{\gamma}\, hyp.\ log. \left(\frac{p_1}{p}\right)}.$$

When the tensions p and p_1 differ little from each other, when $p_1 - p$ is $\backsim \frac{1}{10}\, p$, then we may put:

$$hyp.\ log. \frac{p_1}{p} = hyp.\ log. \left(1 + \frac{p_1 - p}{p}\right) = \frac{p_1 - p}{p}, \text{ and hence}$$

$$v = \sqrt{2\, g \left(\frac{p_1 - p}{\gamma}\right)}.$$

But the height of an external column of air which is in equilibrium by its weight with the pressure $p_1 - p$ (§ 294), is $h = \frac{p_1 - p}{\gamma}$; hence we may put the velocity of efflux $v = \sqrt{2\, g\, h}$, and a perfect analogy with the efflux of water will hereby subsist. For high pressure this formula is not of course sufficient, for in this case:

$$hyp.\ log. \left(\frac{p_1}{p}\right) = \frac{p_1 - p}{p} - \frac{1}{2} \left(\frac{p_1 - p}{p}\right)^2 \text{ at least.}$$

Hence, then, more accurately:

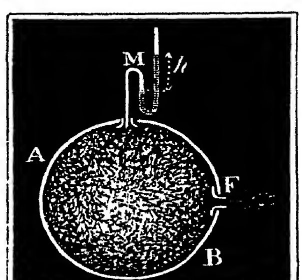
$$v = \sqrt{2\, g \left(\frac{p_1 - p}{\gamma} - \frac{1}{2}\, \frac{(p_1 - p)^2}{p\, \gamma}\right)} = \sqrt{2\, g \left(1 - \frac{p_1 - p}{2\, p}\right) h},$$

or if we represent the height of the barometer by b, $p = b\, \gamma$, and

$$v = \sqrt{2\, g \left(1 - \frac{h}{2\, b}\right) h} = \left(1 - \frac{h}{4\, b}\right) \sqrt{2\, g\, h}.$$

If the discharging orifice F of the vessel AB, Fig. 486, is accurately and smoothly rounded, the particles of air then flow in parallel lines, and hence the quantity of air flowing through the orifice in each second, and measured by the height of the external barometer, is:

Fig. 486.



$$Q = Fv = F \left(1 - \frac{h}{4\, b}\right) \sqrt{2\, g\, h},$$

or more accurately:

$$= F \sqrt{2\, g\, b\ hyp.\ log. \left(\frac{b + h}{b}\right)}.$$

§ 355. The above formulæ do not admit of direct application, because we cannot measure the internal or the external pressure by the

length $b+h$, and b of the columns of air. These pressures are generally measured by columns of water or mercury. As regards the quotient $\frac{p_1}{p} = \frac{b+h}{b}$, it is immaterial whether b and h be expressed in columns of air, water, or mercury, because each reduction of b and h leaves the fraction $\frac{b+h}{b}$ constant, except that the quotient $\frac{p}{\gamma} = b$, is still dependent on the temperature of the effluent air, and varies for different kinds of gas. For atmospheric air (§ 301), if p represent the pressure of air on one square centimetre, and γ the weight of a cubic metre of air, and t the temperature in degrees centigrade, we have

$$\frac{p}{\gamma} = \frac{1+0,00367 \,.\, t}{1,2572},$$ on the other hand, for steam,

$$\frac{p}{\gamma} = \frac{1+0,00367\, t}{0,7857}.$$

If we substitute these values in the general formula for v, we shall obtain for atmospheric air:

$$v = 395 \sqrt{(1+0,00367 \,.\, t)\; hyp.\ log. \left(\frac{b+h}{b}\right)} \text{ metres,}$$

or $\frac{h}{b}$ being small:

$$v = 395 \sqrt{(1+0,00367 \,.\, t)\frac{h}{b}} \text{ metres, and for steam}$$

$$v = 500,6 \sqrt{(1+0,00367 \,.\, t)\; hyp.\ log. \left(\frac{b+h}{b}\right)} \text{ metres.}$$

The theoretical discharge as estimated under the external pressure is $Q = Fv$, but if this is to be estimated at the internal pressure, we must then make $Q_1 p_1 = Q\, p$, hence $Q_1 = \frac{p}{p_1}\, Q = \frac{b\, Q}{b+h}$. Reduced to the temperature of zero, the quantity discharged is:

$$Q_2 = \frac{Q}{1+0,00367 \,.\, t},$$ therefore, for atmospheric air

$$\doteq 395\, F \sqrt{\frac{hyp.\ log.\ (b+h) - hyp.\ log.\ b}{1+0,00367 \,.\, t}} \text{ cubic metres.}$$

If equal masses of air of different temperatures issue from different orifices F and F_1 at the same tension, we then have:

$$\frac{F_1}{F} = \sqrt{\frac{1+0,00367\, t_1}{1+0,00367\, t}}.$$

If, for example, $t = 0$ and $t_1 = 150°$ C., we then have:

$$F_1 = \sqrt{1,5505} \,.\, F = 1,245\, F.$$

If, therefore, a blast furnace is to be supplied with heated air of 150°, we must apply nozzle pipes, which have a one-fourth greater transverse section at the discharging orifice than if cold air were to be used.

For Prussian measure, and centigrade scale of temperature:

$$v = 1258 \,.\, \sqrt{(1 + 0{,}00367\, t)\; hyp.\, log. \left(\frac{b+h}{b}\right)}, \text{ and for steam}$$

$$v = 1595 \,.\, \sqrt{(1 + 0{,}00367\, t)\; hyp.\, log. \left(\frac{b+h}{b}\right)}.$$

For English measure, and Fahrenheit's scale of temperature:

$$v = 1295 \,.\, \sqrt{(1 + 0{,}00204\, t)\; hyp.\, log. \left(\frac{b+h}{b}\right)}, \text{ and for steam}$$

$$v = 1642 \,.\, \sqrt{(1 + 0{,}00204\, t)\; hyp.\, log. \left(\frac{b+h}{b}\right)}.$$

Example. In a large reservoir, air at 120° C., temperature is enclosed, which corresponds to the height of a mercurial manometer of 5 inches, whilst the external barometer stands at 27,2 inches; what quantity of air will flow from this through a round aperture $1\frac{1}{2}$ inch wide? It is:

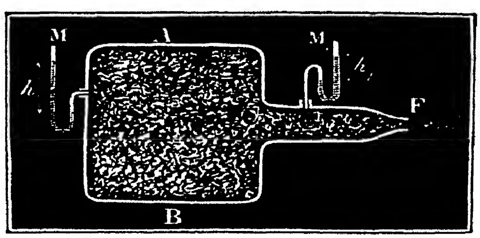
$hyp.\ log. \left(\frac{b+h}{b}\right) = hyp.\ log. \left(\frac{32{,}2}{27{,}2}\right) = hyp.\ log.\ 32{,}2 - hyp.\ log.\ 27{,}2 = 5{,}77455 - 5{,}60580 = 0{,}16875$, hence the velocity of efflux is:

$v = 1258 \,.\, \sqrt{(1 + 0{,}00367 \,.\, 120)\, 0{,}16875} = 1258 \,.\, \sqrt{1{,}4404 \,.\, 0{,}16875} = 620{,}2$ Prussian feet. Now the area of the orifice $= \frac{\pi}{4}\left(\frac{1}{8}\right)^2 = \frac{\pi}{256} = 0{,}01227$ square feet; hence it follows that the discharge $Q = 0{,}01227 \,.\, 620{,}2 = 7{,}61$ cubic feet. Estimated at the interior pressure, it is $= \frac{272}{322} \,.\, 7{,}61 = 6{,}43$ cubic feet, and reduced to the mean height of the barometer, 28 inches and 0° temperature (30 English inches and 32° temperature F.), the quantity discharged is:

$= 7{,}61 \,.\, \frac{272}{280} \,.\, \frac{1}{1{,}4404} = 5{,}13$ cubic feet.

§ 356. *Efflux of Air in Motion.*—The formula of efflux given: suppose the pressure p_1 or the height of the manometer h to be measured at a place where the air is at rest, or has a very slight motion, but if p_1 or h_1 is measured at a place where the air is in motion, if, for instance, the manometer M_1 communicates with the air in a conducting tube CF, Fig. 487, we shall then have to take into account the *vis viva* of the arriving air. If now c be the velocity of the air passing the orifice of the manometer we shall accordingly have to make:

Fig. 487.



$$Q\gamma \,.\, \frac{v^2}{2g} = Q\gamma \,.\, \frac{c^2}{2g} + Q\, p\; hyp.\, log. \left(\frac{p_1}{p}\right),$$

or if F be the transverse section of the orifice, and G that of the tube, or of the air passing the orifice of the manometer, according to the law of Mariotte, $\frac{Gc}{Fv} = \frac{p}{p_1}$, or $Gcp_1 = Fvp$, therefore,

$$c = \frac{F}{G} \,.\, \frac{p}{p_1}\, v,\quad Q\gamma \left[1 - \left(\frac{F}{G}\right)^2 \left(\frac{p}{p_1}\right)^2\right] \frac{v^2}{2g} = Qp\; hyp.\, log. \left(\frac{p_1}{p}\right),$$

and the velocity of efflux in question:

$$v = \frac{\sqrt{2\, g \frac{p}{\gamma}\, hyp.\, log. \left(\frac{p_1}{p}\right)}}{\sqrt{1 - \left(\frac{Fp}{G\, p_1}\right)^2}}.$$

The velocity of efflux is, therefore, here exactly like that of water from vessels, the velocity is greater, the greater the ratio $\frac{F}{G}$ of the transverse section of the orifice to that of the tube or the arriving current of air. From this it is evident, that under otherwise similar circumstances, the height of the manometer p_1 is so much the less the narrower the conducting tube is, or the greater the velocity of the air issuing from it.

Examples.—1. A mercurial manometer, placed upon an air tube $3\frac{1}{2}$ inches wide, stands at $2\frac{1}{2}$ inches, while the air flows from its conical extremity through a round orifice 2 inches in diameter; with what velocity will the current move? If the external barometer stand at $27\frac{1}{2}$ inches, we shall then have $\frac{p_1}{p} = \frac{27\frac{1}{2} + 2\frac{1}{2}}{27\frac{1}{2}} = \frac{30}{27,5} = \frac{12}{11}$ and $\frac{F}{G}\frac{p}{p_1} = \left(\frac{2}{3,5}\right)^2 \cdot \frac{11}{12} = \frac{16\,.\,11}{49\,.\,12} = \frac{44}{147}$; hence the theoretical velocity of efflux at a temperature of the air 10° C.:

$$v = \frac{1258\,.\,\sqrt{1,0367\,.\, hyp.\, log. \left(\frac{12}{11}\right)}}{\sqrt{1 - \left(\frac{44}{147}\right)^2}} = \frac{1258\,\sqrt{1,0367\,.\,0,087}}{\sqrt{0,9104}} = 396 \text{ Prussian feet.—}$$

2. The tension p_2 in the air regulator, where the air is without motion, is given by the formula,

$$hyp.\, log. \left(\frac{p_2}{p}\right) = \frac{v^2}{2g} \cdot \frac{\gamma}{p}, \text{ or } hyp.\, log.\, p_2 = hyp.\, log.\, p + \frac{hyp.\, log. \left(\frac{p_1}{p}\right)}{1 - \left(\frac{Fp}{Gp_1}\right)^2}$$

therefore, in the present case, $= hyp.\, log.\, 27,5 + \frac{0,087}{0,9104} = 3,3142 + 0,0965 = 3,4107.$ Hence it follows that $p_2 = 30,3$ inches.

§ 357. *Efflux under Decreasing Pressure.*—If an air reservoir has no influx, whilst an uninterrupted efflux goes on, the density and tension gradually diminish, and hence the velocity of efflux becomes less and less. We may determine in the following manner in what ratio this diminution is to the time and to its discharge.

Let V be the volume of the reservoir, h_0 the initial height of the manometer, and h_n the height of the manometer at the end of a certain time t, b the height of the external barometer. Then the quantity of air in the reservoir at the commencement reduced to the external pressure $= \frac{V(b+h_0)}{b}$, and at the end of the time t, $= \frac{V(b+h_n)}{b}$, and, consequently, the quantity discharged in the time t, and at the external pressure is:

$$V_n = \frac{V(b+h_0)}{b} - \frac{V(b+h_n)}{b} = \frac{V(b_0 - h_n)}{b};$$

and, inversely, the height of the manometer corresponding to the discharge V_n is:

$$h_n = h_0 - \frac{V_n}{V} . b.$$

If we take four intervals, and the initial height of the manometer h_0, and at the end of the time $t = h_4$, and

$$h_1 = h_0 - \frac{h_0 - h_4}{4},\ h_2 = h_0 - \tfrac{2}{4}(h_0 - h_4),\ \text{and}$$

$h_3 = h_0 - \frac{3}{4}(h_0 - h_4)$, we shall then obtain by Simpson's rule the time

$$t = \frac{V(h_0 - h_4)}{12\, Fb\sqrt{2g\frac{p}{\gamma}}}\left(\frac{1}{\sqrt{hyp.\ log.\left(\frac{b+h_0}{b}\right)}} + \frac{4}{\sqrt{hyp.\ log.\left(\frac{b+h_1}{b}\right)}}\right.$$

$$+ \frac{2}{\sqrt{hyp.\ log.\left(\frac{b+h_2}{b}\right)}} + \frac{4}{\sqrt{hyp.\ log.\left(\frac{b+h_3}{b}\right)}}$$

$$\left. + \frac{1}{\sqrt{hyp.\ log.\left(\frac{b+h_4}{b}\right)}}\right).$$

For moderate pressures or heights of the manometer:

$$hyp.\ log.\left(\frac{b+h}{b}\right) = \frac{h}{b}\left(1 - \frac{h}{2b}\right),$$

$$\text{consequently } \sqrt{hyp.\ log.\left(\frac{b+h}{b}\right)} = \left(1 - \frac{h}{4b}\right)\sqrt{\frac{h}{b}} \text{ and}$$

$$\frac{1}{\sqrt{hyp.\ log.\left(\frac{b+h}{b}\right)}} = \left(1 + \frac{h}{4b}\right)\sqrt{\frac{h}{b}}.$$

If we now take n intervals, and therefore the discharge for one interval: $\frac{V_1}{n} = \frac{V(h_0 - h_n)}{nb}$, we shall then obtain the corresponding element of time:

$$\tau = \frac{V(h_0 - h_n)}{nb} \div \mu F\sqrt{2g\frac{p}{\gamma}\, hyp.\ log.\left(\frac{b+h}{b}\right)}$$

$$= \frac{\frac{V(h_0 - h_n)}{nb}\left(1 + \frac{h}{4b}\right)\sqrt{\frac{b}{h}}}{F\sqrt{2g\frac{p}{\gamma}}}$$

$$= \frac{V(h_0 - h_n)\left(h^{-\frac{1}{2}} + \frac{h^{\frac{1}{2}}}{4b}\right)}{n \cdot F\sqrt{2gb\frac{p}{\gamma}}}.$$

Now if we substitute for h; h_0, h_1, h_2, ... h_n, we shall then obtain the sum of all the

$$\left(\frac{h_0-h_n}{n}\right) h^{-\frac{1}{2}} = 2(h_0^{\frac{1}{2}}-h_n^{\frac{1}{2}}) = 2(\sqrt{h_0}-\sqrt{h_n}).$$

and the sum of all the

$$\left(\frac{h_0-h_n}{n}\right) h^{\frac{1}{2}} = \tfrac{2}{3}\left(h_0^{\frac{3}{2}}-h_n^{\frac{3}{2}}\right) = \tfrac{2}{3}(\sqrt{h_0^3}-\sqrt{h_n^3}),$$

whence the sum of all the small intervals of time, or the whole time in which h_n passes into h_0, and the quantity of air

$V_n = \frac{V(h_0-h_n)}{b}$ which flows out, is:

$$t = \frac{2V}{F\sqrt{2gb\frac{p}{\gamma}}}\left[(\sqrt{h_0}-\sqrt{h_n}) + \frac{1}{12b}(\sqrt{h_0}^3-\sqrt{h_n}^3)\right], \text{ or}$$

$$= \frac{2V}{F\sqrt{2gb\frac{p}{\gamma}}}(\sqrt{h_0}-\sqrt{h_n})\left(1+\frac{h_0+\sqrt{h_0h_n}+h_n}{12b}\right),$$

approximately:

$$= \frac{2V}{F\sqrt{2gb\frac{p}{\gamma}}}(\sqrt{h_0}-\sqrt{h_n})\left(1+\frac{h_0+h_n}{8b}\right).$$

Example. A 50 feet long and 5 feet wide cylindrical air-regulator of a blowing machine is filled with air; the height of its manometer $h = 10$ inches, and the thermometer stands at 6° C. If now a flow of air takes place in a space where the height of the barometer is 27 inches, through a 1-inch wide circular orifice, then the question arises, in what time will the height of the manometer fall to 7 inches, and what will be the corresponding discharge? The volume of the chamber is for Prussian measures,— $= \frac{\pi}{4} \cdot 5^2 \cdot 50 = 1250 \cdot \frac{\pi}{4} = 981{,}75$ cubic feet, hence the discharge, measured at the external pressure, is $V_1 = \left(\frac{h_0-h_n}{b}\right) V = \left(\frac{10-7}{27}\right) \cdot 981{,}75 = 109{,}08$ cubic feet.

Now $\sqrt{2g\frac{p}{\gamma}} = 1258\sqrt{1+0{,}00367 \cdot t} = 1258\sqrt{1{,}02202} = 1272$, and $F = \frac{\pi}{4}(\frac{1}{12})^2 = \frac{\pi}{576} = 0{,}005454$ square feet, hence the time of efflux in question is

$$t = \frac{2 \cdot 981{,}75}{0{,}005454 \cdot 1272}\left(\sqrt{\frac{10}{27}}-\sqrt{\frac{7}{27}}\right)\left(1+\frac{10+7}{8 \cdot 27}\right)$$

$$= \frac{1963{,}5}{5{,}454 \cdot 1{,}272} \cdot 0{,}0994 \cdot 1{,}079 = 30{,}3 \text{ seconds.}$$

§ 358. *Co-efficients of Efflux.*—The phenomena of contraction, which we have considered in the efflux of water from vessels, occur also in the efflux of air. If the orifice of efflux be cut in a thin plate, the air passing through it has a smaller transverse section than the orifice, and on this account the discharge is less than the product Fv of the transverse section F of the orifice and the theoretical velocity v. Let $\frac{F_1}{F}$ be the ratio of the transverse section F_1 of the blast to that of the orifice F, $= \mu$, we then have the effective discharge as for water:

$$Q_1 = \mu\, Q = F_1 v = \mu\, F v = \mu\, F \sqrt{2g\,\frac{p}{\gamma}\, hyp.\ log. \left(\frac{p_1}{p}\right)}.$$

From the author's reduction of Koch's experiments at pressures of the manometer of from $\frac{1}{200}$ to $\frac{1}{5}$ of an atmosphere, we may take the mean of $\mu = 0{,}58$.

The effective discharge in the issuing of air through short cylindrical adjutages, is likewise less than that determined theoretically; we have, therefore, to multiply this latter by a number deduced from experiment, the co-efficient of efflux, μ in order to obtain the former; only here μ is not the ratio of the transverse section $\frac{F_1}{F}$, but the ratio $\frac{v_1}{v}$ of the effective velocity of efflux v_1 to the theoretical v. Koch's experiments give for the above pressures, in the flow of air through cylindrical adjutages, which were nearly all six times as long as wide, as a mean $\mu = 0{,}74$.

Conically convergent adjutages, similar to the nozzles of bellows, give a still greater co-efficient of efflux; a tube of 6° lateral convergence in the experiments of Koch, gave when five times as long as wide, the mean co-efficient $\mu = 0{,}85$.

From this, therefore, the effective discharge for the flow of air through orifices in a thin plate, measured at the external pressure, is

$$Q_1 = 751{,}1\, F \left(1 - \frac{h}{4b}\right) \sqrt{(1 + 0{,}00367\, t)\, \frac{b}{h}} \text{ cubic feet (Eng.),}$$

for efflux through short cylindrical adjutages:

$$Q_1 = 958{,}3\, F \left(1 - \frac{h}{4b}\right) \sqrt{(1 + 0{,}00367\, t)\, \frac{h}{b}} \text{ cubic feet,}$$

and through conical adjutages of 6° convergence.

$$Q_1 = 1090{,}7\, F \left(1 - \frac{h}{4b}\right) \sqrt{(1 + 0{,}00367\, t)\, \frac{h}{b}} \text{ cubic feet.*}$$

* Experiments on the efflux of air have been undertaken by Young, Schmidt, Lagerhjelm, Koch, d'Aubuisson, Buff, and in later time, by Pecqueur, Saint-Venant, and Wantzel. For an account of the experiments of Young and Schmidt, we may refer to Gilbert's "Annalen," vol. 22, 1801, and vol. 6, 1820, and to Poggendorff's "Annalen," vol. 2, 1824; for those of Koch and Buff, to the "Studien des göttingschen Vereines bergmännischer Freunde," vol. 1, 1824; vol. 3, 1833; vol. 4, 1837, and vol. 5, 1838; also in Poggendorff's "Annalen," vol. 27, 1836, and vol. 40, 1837. The experiments of Lagerhjelm are described in the Swedish work, "Hydrauliska Försök of Lagerhjelm, Forselles och Kallstenius," 1 vol. Stockholm, 1818. D'Aubuisson's experiments are to be found in the "Annales des Mines," vol. 11, 1825; vol. 13, 1826; vol. 14, 1827, and likewise in his "Traité d' Hydraulique." The latest experiments instituted in France are reported in the "Polytechnischen Centralblatt," vol. 6, 1845. Most of these experiments were made with very narrow orifices, and, therefore, scarcely answer the purpose in practice. The experiments of d'Aubuisson and Koch deserve most consideration; and next to them, perhaps, those of Pecqueur; but the most extensive are those of Koch. The wished-for accordance is hardly to be met with in the results of all these experiments; the co-efficients of efflux found by d'Aubuisson vary considerably from those calculated by Koch. The grounds for my placing the most confidence in the co-efficients of Koch, are given in the "Allgemeinen Maschinenencyclopädie," under the article "Ausfluz," and in a Memoir of mine in Poggendorff's "Annalen," vol. 51, 1840. [For calculations of the above, and all similar cases, the co-efficient of t for the Fahrenheit's thermometer is 0,002039 instead of 0,00367; (see above, p. 346;) but the degrees computed are actually $t - 32$ on that scale.]—Am. Ed.

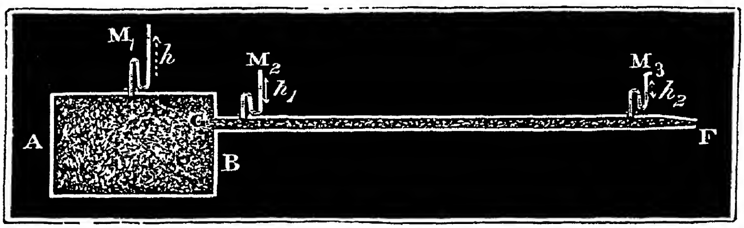
Example. If the two orifices of a bellows together possess an area of 3 square inches, if, further, the pressure of the manometer is 3 inches, the external barometer $27\frac{1}{2}$ inches, and the temperature of the air 15°, then is the discharge:

$$Q_1 = 1069 \cdot \tfrac{3}{144} \left(1 - \frac{3}{4 \cdot 27,5}\right) \sqrt{(1 + 0,00367 \cdot 15)\, \frac{3}{27,5}}$$

$$= 22,27 \cdot \frac{107}{110} \sqrt{1,055 \cdot \tfrac{6}{55}} = 21,66 \sqrt{0,1151} = 7,34 \text{ cubic feet.}$$

§ 359. *Flow through Tubes.*—If the air issues through a long tube GF, Fig. 488, it has then the resistance of friction to overcome in

Fig. 488.



the same manner as water; this resistance may also be measured by the height of a column of air, which has for expression $h_n = \zeta \cdot \frac{l}{d} \cdot \frac{v^2}{2g}$, where, as in the conducting of water, v represents the velocity, l the length, d the width of the tube, and ζ a co-efficient of resistance to be determined by experiment.

Numerous experiments of Girard, d'Aubuisson, Buff and Pecqueur, lead to the mean value $\zeta = 0,024$. From this, therefore, the resistance generated by the friction of air in tubes may be measured by the height $h_n = 0,024 \frac{l}{d} \cdot \frac{v^2}{2g}$ of a column of air, or by the height $h_n = 0,0000023 \frac{l}{d} \cdot \frac{v^2}{2g}$ of a column of quicksilver, and the manometer will stand at this much less height at the end of the conducting tube than at the beginning.

If at the end of a conducting tube of the width d, the manometer stands at h_2, whilst the air flows through an orifice of the width d, then from what precedes, the velocity of discharge will be:

$$v = \frac{\sqrt{2\,g\,\frac{p}{\gamma}\, hyp.\, log.\left(\frac{b + h_2}{b}\right)}}{\sqrt{1 - \left(\frac{b}{b + h_2}\right)^2 \left(\frac{d_1}{d}\right)^4}};$$

but if h_1 be the height of the manometer at the beginning of the conduit, we shall then have:

$$\frac{p}{\gamma}\, hyp.\, log.\left(\frac{b - h_1}{b}\right) = \left[1 + \left(\frac{b}{b + h_1}\right)^2 \left(\frac{d_1}{d}\right)^4 + 0,024\, \frac{l}{d} \left(\frac{d_1}{d}\right)^4\right] \frac{v^2}{2g},$$

because the velocity in the tube $= \frac{d_1^{\,2}}{d^2}\, v$; hence in this case

$$v = \frac{\sqrt{2\,g\,\frac{p}{\gamma}\;hyp.\,log.\left(\frac{b+h_1}{b}\right)}}{\sqrt{1+\left[0{,}024\frac{l}{d}+\left(\frac{b}{b+h_1}\right)^2\right]\left(\frac{d_1}{d}\right)^4}}.$$

If, lastly, the height of the manometer h is measured in the reservoir at the beginning of the conduit, where the air may be regarded as at rest, we then have:

$$v = \frac{\sqrt{2\,g\,\frac{p}{\gamma}\;hyp.\,log.\left(\frac{b+h}{b}\right)}}{\sqrt{1+0{,}024\,\frac{l\,d_1^4}{d^5}}}.$$

If, further, we put the co-efficient of resistance ζ for entrance into the tube, which when $\mu_1 = 0{,}74$ amounts to 0,826, and, further, join to it the co-efficient of efflux μ for the outer adjutage, we then obtain for the velocity:

$$v = \frac{\mu\sqrt{2\,g\,\frac{p}{\gamma}\;hyp.\,log.\left(\frac{b+h}{b}\right)}}{\sqrt{1+\zeta+0{,}024\,\frac{l\,d_1^4}{d^5}}}$$

$$\text{or} = \frac{1294\,\mu\sqrt{(1+0{,}00367\,t)\;hyp.\,log.\left(\frac{b+h}{b}\right)}}{\sqrt{1+\zeta+0{,}024\,\frac{l\,d_1^4}{d^5}}}\ \text{feet (Pruss.)}$$

According as the point of the interior orifice lies s lower or higher than the point of the exterior orifice, we have to add $\pm s$ to the quantity under the radical in the denominator. Moreover, other hindrances may present themselves in the tube, such as curvatures, contractions, and widenings, &c. Satisfactory experiments on these obstacles do not exist, but we may assume with great probability that these resistances are not much different from what takes place in the case of water, because the co-efficients of efflux, and the co-efficient of friction are nearly the same for air as for water.

As long, therefore, as no further experiments are made on this subject, we may avail ourselves with tolerable safety of the co-efficient of resistance found for water in investigations on the motion and flow of air.

Example. In the regulator at the head of a 320 feet long and 4 inch wide air-conductor, the mercurial manometer stands at 3,1 inch, whilst the external barometer is at 27,2 inch; further, the width of the orifice of the conically contracted extremity of the conductor is 2 inches, and the temperature of the air 20° C., what quantity of air will this conductor deliver? It will be:

$1 + \zeta + 0{,}024\,\frac{l d_1^4}{d^5} = 1{,}826 + 0{,}024 \cdot \frac{320}{\frac{1}{3}} \cdot (\tfrac{2}{4})^4 = 1{,}826 + 0{,}024 \cdot \frac{320 \cdot 3}{16} = 1{,}826 + 1{,}44 = 3{,}266$; further, $(1 + 0{,}00367\,t)\,hyp.\,log.\left(\frac{b+h}{b}\right)$,

$= (1 + 0{,}00367 \,.\, 20)\; hyp.log. \left(\frac{30{,}3}{27{,}2}\right) = 1{,}0734 \,.\, (5{,}7137 - 5{,}6058)$

$= 1{,}0734 \,.\, 0{,}1079 = 0{,}1158$; if now, further, we introduce the co-efficient of efflux, $\mu = 0{,}85$, we shall then obtain the velocity of flow:

$v = \frac{1258 \,.\, 0{,}85 \sqrt{0{,}1158}}{\sqrt{3{,}266}} = 201{,}3$ feet; and lastly, the discharge:

$Q = \frac{\pi d_1^2}{4} \,.\, v = \frac{\pi}{4} \,.\, \frac{201{,}3}{36} = 4{,}39$ cubic feet (Prussian).

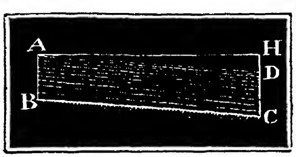
CHAPTER VII.

ON THE MOTION OF WATER IN CANALS AND RIVERS.

§ 360. *Running Water.*—The doctrine of the motion of water in canals and rivers, forms the second main division of hydraulics. Water flows either in a natural or in an artificial bed. In the first case, it forms streams, rivers, brooks; in the second, canals, cuts, drains, &c. In the theory of the motion of flowing water, this distinction is of little moment.

The bed of a river consists of the *bottom* and the two *banks* or *shores*. The *transverse section* is obtained by a plane at right angles to the direction of motion of the flowing water. Its *perimeter* is that of the transverse section, which again consists of the *air* and the *water section*. A vertical plane in the direction of the flowing water gives the *longitudinal section* or *profile*. By the *slope* or *declivity* of flowing water is understood the angle of inclination of its surface to the horizon. The *fall*, which is the vertical distance of the two extreme points of a definite length of the fluid surface, serves to assign the angle for a definite length of the flowing stream. For the length of course, $AD = l$, Fig. 489, BC is the bottom of the channel, $DH = h$ the fall, and the angle $DAH = \delta$, the slope $sin.\ \delta = \frac{h}{l}$ = absolute fall per unit of length.*

Fig. 489.



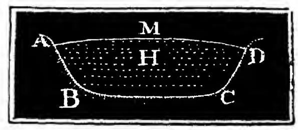
§ 361. *Different Velocities in the Transverse Section.*—The velocity of water in one and the same transverse section is very different at

* The fall of brooks and rivers is very various. The Elbe, for example, for the extent of a German mile from the Upper Elbe to Podiebrad, has a fall of 57 feet, from thence to Leitmeritz 9 feet, from there to Mühlberg a mean of 5,8, and from thence to Magdeburg 2,5 feet. Mountain brooks have a fall of from 40 to 400 feet per German mile. For further particulars, *see* "Vergleichende hydrographische Tabellen," &c., von Stranz. Canals and other artificial water conduits have much smaller falls. Here the absolute fall, at most, is 0,001, often 0,0001, and even less. More on this subject will be given in the Second Part.

different points. The adhesion of the water to its bed, and the cohesion of the particles to each other, cause those lying nearer to the sides of the bed to suffer some constraint in their motion, and hence, to flow more slowly than the more remote. For this reason the velocity diminishes from the surface downwards to the bed, and is least near the side or at the bottom. The greatest velocity is found for straight rivers, generally in the middle, or at that part of the free surface of the water where there is the greatest depth. The place where the water attains its maximum velocity is called the *line of current*, and the deepest part of the bed, the *mid-channel*.

The upper surface does not form an exact horizontal line, because the elements lying on the surface of water, flow on with different velocities with respect to each other, they therefore exert on each other different pressures; the quicker ones a less, and the slower a greater pressure, and thus for the maintenance of relative equilibrium, the quicker elements superpose themselves on the slower. If v and v_1 are the velocities of two elements M and A, Fig. 490, then according to the doctrine of hydraulic pressure (§ 307) the difference of level of the two elements is:

Fig. 490.



$$MH = h = \frac{v^2}{2g} - \frac{v_1^{\,2}}{2g} = \frac{v^2 - v_1^{\,2}}{2g}.$$

This difference of level is always very small. If, for example, $v_1 = 0{,}9\ v$, and $v = 5$ feet, we then have this

$$= (1 - 0{,}81)\,\frac{v^2}{2g} = 0{,}19\ .\ 0{,}0155\ .\ 25 = 0{,}0736 \text{ feet} = 0{,}88 \text{ inches}$$

(Eng.). For this reason the water stands highest in the current, and lowest at the banks.

In bends, the current is generally near the concave bank.

§ 362. *Permanent Motion of Water.*—The mean velocity of water in a transverse section is, according to § 308:

$$c = \frac{Q}{F} = \frac{\text{quantity of water per second}}{\text{area of section}}.$$

The mean velocity besides may be further calculated from the velocities c_1, c_2, c_3, &c., of the separate portions of the section, and from the areas F_1, F_2, F_3, &c. It is namely:

$$Q = F_1 c_1 + F_2 c_2 + F_3 c_3 + \ldots,$$

and hence also:

$$c = \frac{F_1 c_1 + F_2 c_2 + \ldots}{F_1 + F_2 + \ldots}.$$

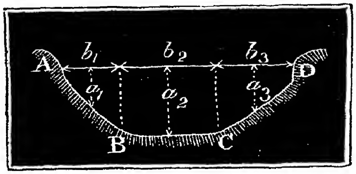
Besides the mean velocity, the mean depth of water has to be introduced, that is, the depth a which a section must have at all points that it may have the same area as it actually has with the variable depths a_1, a_2, a_3, &c. Hence, therefore,

$$a = \frac{F}{b} = \frac{\text{area of section}}{\text{breadth of section}}.$$

If the separate parts of the breadth b_1, b_2, b_3, have the corresponding mean depths a_1, a_2, a_3, &c., Fig. 491, we then have:

$$F = a_1 b_1 + a_2 b_2 + \dots,$$

Fig. 491.



and hence also:

$$a = \frac{a_1 b_1 + a_2 b_2 + \dots}{b_1 + b_2 + \dots}.$$

Lastly:

$$c = \frac{a_1 b_1 c_1 + a_2 b_2 c_2 + \dots}{a_1 b_1 + a_2 b_2 + \dots},$$

and if the portions b_1, b_2, &c., be of equal size,

$$c = \frac{a_1 c_1 + a_2 c_2 + \dots}{a_1 + a_2 + \dots}.$$

A river or brook is in a state of *permanency* when an equal quantity of water flows through each of its transverse sections in an equal time; when, therefore, Q or the product Fc of the area of the section and the mean velocity throughout the whole extent of the stream is a constant number. Hence this simple law comes out: *in the permanent motion of water, the mean velocities in two transverse sections are to each other inversely as the areas of these sections.*

Examples.—1. At the section of a canal, $ABCD$, Fig. 491, it was found that the

Portions of the breadth - - -	$b_1 = 3{,}1$ feet,	$b_2 = 5{,}4$ feet,	$b_3 = 4{,}3$ feet
Mean depth - - - - - -	$a_1 = 2{,}5$ "	$a_2 = 4{,}5$ "	$a_3 = 3{,}0$ "
Corresponding mean velocities - -	$c_1 = 2{,}9$ "	$c_2 = 3{,}7$ "	$c_3 = 3{,}2$ "

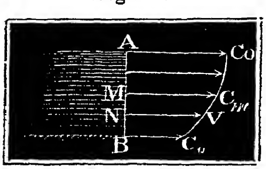
Hence the area of these profiles $F = 3{,}1 \,.\, 2{,}5 + 5{,}4 \,.\, 4{,}5 + 4{,}3 \,.\, 3{,}0 = 44{,}95$ square feet, and the discharge:
$Q = 3{,}1 \,.\, 2{,}5 \,.\, 2{,}9 + 5{,}4 \,.\, 4{,}5 \,.\, 3{,}7 + 4{,}3 \,.\, 3{,}0 \,.\, 3{,}2 = 153{,}665$ cubic feet, and the mean velocity $c = \frac{Q}{F} = \frac{153{,}665}{44{,}95} = 3{,}419$ feet.

2. When a cut is to conduct 4,5 cubic feet of water with a mean velocity c of 2 feet, we must then give to it a transverse section of $\frac{4{,}5}{2} = 2{,}25$ square foot area.—3. If one and the same stream has a mean velocity of $2\frac{1}{4}$ feet at a place 560 feet broad and 9 feet mean depth, it will then have, at a place 320 feet broad and 7,5 feet mean depth, the mean velocity

$$c = \frac{560 \,.\, 9}{320 \,.\, 7{,}5} \,.\, 2{,}25 = \frac{567}{120} = 4{,}725 \text{ feet.}$$

§ 363. *Mean Velocity.*—If we divide the depths of water at any point of a flowing stream into equal parts, and raise ordinates upon them corresponding to the velocities, we shall then obtain a scale of the velocity of the current AB, Fig. 492. Although it may be granted that the law of this scale, or of the difference of velocity is expressed by some curve, as according to Gerstner by an ellipse, yet it is allowable, without fear of any great error, to substitute for this a straight line, or assume that the velocity diminishes uniformly with the depth, because the diminution of velocity downwards is always very small. From the experiments of Ximenes, Brünnings, and Funk, the mean velocity in a perpendicular $c_m = 0{,}915\, c_0$, where c_0 repre-

Fig. 492.



sents the velocity at the surface, or the maximum velocity. The velocity, therefore, diminishes from the surface to the middle $\mathcal{M}$

$$\text{by } c_0 - c_m = (1 - 0{,}915)\, c_0 = 0{,}085\, c_0,$$

and, consequently, the velocity below or at the foot of the perpendicular may be put

$$c_u = c_0 - 2\,.\,0{,}085\, c_0 = (1 - 0{,}170)\, c_0 = 0{,}83\, c_0.$$

If, now, the whole depth $= a$, we then have, by assuming a straight line for the scale of the velocities, the corresponding velocity for a depth $\mathcal{AN} = x$, below the water

$$v = c_0 - (c_0 - c_u)\,\frac{x}{a} = \left(1 - 0{,}17\,\frac{x}{a}\right) c_0.$$

Further, let $c_0, c_1, c_2, \ldots$ be the superficial velocities of a whole transverse profile of not very variable depth, we have then the corresponding velocities at a mean depth: $0{,}915\, c_0$, $0{,}915\, c_1$, $0{,}915\, c_2$, and hence the mean velocity in the whole profile:

$$c = 0{,}915\,\frac{(c_0 + c_1 + c_2 + \ldots c_n)}{n}.$$

Lastly, if we assume that the velocity diminishes from the line of current towards the banks, as it does according to the depth, we may then again put the mean superficial velocity

$$\frac{(c_0 + c_1 + \ldots + c_n)}{n} = 0{,}915\, c_0,$$

and so obtain the mean velocity in the whole profile:

$$c = 0{,}915\,.\,0{,}915\,.\,c_0 = 0{,}837\,.\,c_0,$$

i. e. from 83 to 84 per cent. of the maximum velocity, or of that of the line of current.

Prony deduced from Du Buat's experiments conducted with very small channels, and for these cases perhaps more correctly:

$$c_m = \left(\frac{2{,}372 + c_0}{3{,}153 + c_0}\right) c_0 \text{ metre} = \left(\frac{7{,}71 + c_0}{10{,}25 + c_0}\right) c_0 \text{ feet English.}$$

For medium velocities of 3 feet it hence follows that $c_m = 0{,}81\, c_0$.

Example. In the line of current of a brook the velocity of the water is 4 feet, and the depth 6 feet, we have then the mean velocity at a corresponding perpendicular $c_m = 0{,}915\,.\,4 = 3{,}66$ feet, and that at the bottom $= 0{,}83\,.\,4 = 3{,}32$ feet; further, the velocity 2 feet below the surface is $v = (1 - 0{,}17\,.\,\frac{2}{6})\,4 = (1 - 0{,}057)\,4 = 3{,}772$ feet; lastly, the mean velocity throughout the profile is, $c = 0{,}837\,.\,4 = 3{,}348$ feet, and according to Prony, $c = \frac{11{,}50}{13{,}97}\,.\,4 = \frac{46}{13{,}97} = 3{,}29$ feet.*

§ 364. *The Best Form of Transverse Sections.*—The resistance which the bed opposes to the motion of the water in virtue of its adhesion, viscosity, or friction, increases with the surface of contact between the bed and the water, and therefore with the perimeter p of the water profile, or of the portion of the transverse section which comprises the bed. But as more filaments of water pass through a

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* This and the following subjects have been fully treated of under the article "Bewegung des Wassers," in the "Allgemeinen Maschinenencyclopädie." New experiments and new views may be found in the following writings: Lahmeyer's "Erfahrungsresultate über die Bewegung des Wassers in Fluszbetten und Kanälen." Brunswick, 1845.

profile, the greater its area is, so this resistance of a filament increases also inversely as the area, and hence on the whole as the quotient $\frac{p}{F}$ of the perimeter of the water profile, and the area of the whole transverse profile.

That the resistance of friction of a running stream or river may be the smallest possible, we must give to its transverse section that form for which the perimeter p for a given area is a minimum, or the area for a given perimeter a maximum. In enclosed conduits, as, for example, pipes, p is the entire perimeter of the figure formed by the transverse profile. Now of all figures having an equal number of sides, the regular figure, and again, of all regular figures that which has the greater number of sides, has for the same area the least perimeter; hence for enclosed conduits, the co-efficient of friction comes out the less, the nearer its transverse profile approaches to a regular figure, and the greater its number of sides; and the circle, which is a regular figure of an infinite number of sides, is in this case the profile which corresponds to the minimum of friction. We must, therefore, in estimating this resistance of friction, leave out of our consideration in the quotient $\frac{p}{F}$ the upper side or surface in contact with the air. The rectangular and trapezoidal sections are those generally applied to canals, cuts, water-courses, &c. A horizontal line EF, Fig. 493, passing through the centre M of the square AC, divides as well the area as also the perimeter into two equal parts, hence it follows that what is true for the square is also correct for these halves, and, accordingly, of all rectangular transverse profiles, the half square AE, or that which is twice as broad as it is deep, corresponds to the least resistance of friction. The regular hexagon ACE, Fig. 494, may be likewise divided by a horizontal line CF into two equal trapeziums, each of which, like the entire hexagon, has the greatest relative area, and, consequently, of all trapezoidal profiles, half the regular hexagon or the trapezium $ABCF$ with the angle of slope $AFM = BCM$ of 60° is that

Fig. 493.

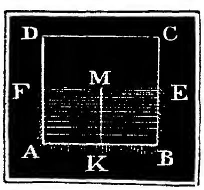


Fig. 494.

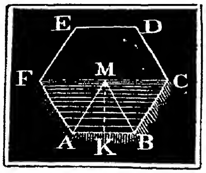


Fig. 495.

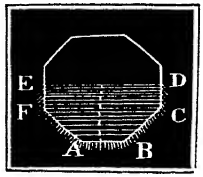
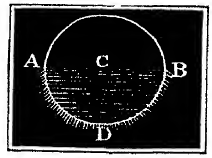


Fig. 496.

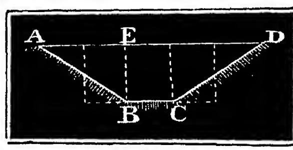


which, when applied, gives the least resistance of friction. Half the regular octagon ADE, Fig. 495, half the regular decagon, and, lastly, the semi-circle ADB, Fig. 496, afford under given circumstances

the most advantageous transverse profiles for canals. The trapezoidal, or half the regular hexagon, gives a still less resistance than half the square or rectangle, the ratio of whose sides is 1 to 2, because the hexagon has a less relative perimeter than the square. Half the regular decagon gives a still less friction, and, in general, the minimum of friction corresponds to the semi-circle. The profiles of channels of wood, stone or iron only, are made semi-circular and rectangular; the profiles of canals, on the other hand, which are cut and bricked, are constructed of the trapezoidal figure. Other figures, in consequence of difficulties in the execution, are not easily applicable.

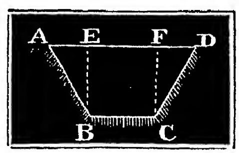
§ 365. In the case where a canal is not walled up, but dug out of loose earth or sand, the angle of 60° slope is too great, and the relative slope *cotg*. 60° = 0,57735 too small, because the banks would not have a sufficient stability; we are, therefore, under the necessity of applying the trapezoidal profile, for which the inclination of the sides to the base must be still less than 60°, perhaps scarcely 45° or even less. For a trapezoidal profile $ABCD$, Fig. 497, which has a perimeter and area equal to that of half the square, the relative slope $= \frac{4}{3}$, and the angle of slope hardly 36° 52′. If the height BE be divided into three equal parts, the base BC will then have two of them, the parallel line AD ten, and each of the sides $AB = CD$ = five parts. In many cases the slope is made = 2, to which belongs an angle of 26° 34′, and sometimes it is even made still greater.

Fig. 497.



In every case the angle of slope $BAE = \odot$, Fig. 498, or the slope $n = \frac{AE}{BE} = cotang. \odot$ may be regarded as a given quantity dependent on the nature of the ground in which the canal is dug, and hence the dimensions of the profile which offers the least resistance have only further to be determined. Let the lower breadth $BC = b$, the depth $BE = a$, and the slope $= n$, we then obtain for the perimeter:

Fig. 498.



$$AB + BC + CD = p = b + 2\sqrt{a^2 + n^2a^2} = b + 2a\sqrt{1 + n^2},$$

for the area:

$$F = ab + naa = a(b + na),$$

and hence, inversely, $b = \frac{F}{a} - na$, and the ratio:

$$\frac{p}{F} = \frac{1}{a} + \frac{a}{F}(2\sqrt{n^2 + 1} - n).$$

If we substitute for a, $a + x$, where x is a small number, we may then put:

$$\frac{p}{F} = \frac{1}{a + x} + \frac{(a + x)}{F}(2\sqrt{n^2 + 1} - n)$$

$$= \frac{1}{a}\left(1 - \frac{x}{a} + \frac{x^2}{a^2}\right) + \frac{a + x}{F}\left(2\sqrt{n^2 + 1} - n\right)$$

$$= \frac{1}{a} + \frac{a}{F}\left(2\sqrt{n^2 + 1} - n\right) + \left(\frac{2\sqrt{n^2 + 1} - n}{F} - \frac{1}{a^2}\right) x + \frac{x^2}{a^3}.$$

Now that this value may be greater not only for a positive, but also for a negative value of x, than the first

$$\frac{1}{a} + \frac{a}{F}\left(2\sqrt{n^2 + 1} - n\right),$$

it is necessary that the member with the factor x should vanish, and therefore that $\frac{p}{F}$ may become a minimum, we must have

$$\frac{2\sqrt{n^2+1} - n}{F} - \frac{1}{a^2} = 0, \textit{ i. e. } a^2 = \frac{F}{2\sqrt{n^2+1} - n},$$

or since:

$$n = cotang.\ \Theta \text{ and } \sqrt{n^2 + 1} = \frac{1}{sin.\ \Theta},\ a^2 = \frac{F\ sin.\ \Theta}{2 - cos.\ \Theta}.$$

Hence, therefore, the most appropriate form of profile corresponding to a given angle of slope Θ, and a given area is determined by

$$a = \sqrt{\frac{F\ sin.\ \Theta}{2 - cos.\ \Theta}} \text{ and } b = \frac{F}{a} - a\ cotang.\ \Theta.$$

Example. What dimensions must be given to the transverse profile of a canal, whose banks are to have 40° slope, and which is to conduct a quantity of water Q of 75 cubic feet, with a mean velocity of 3 feet? $F = \frac{Q}{c} = \frac{75}{3} = 25$ square feet, hence the depth $a = \sqrt{\frac{25\ sin.\ 40^\circ}{2 - cos.\ 40^\circ}} = 5\sqrt{\frac{0,64279}{1,23396}} = 3,609$ feet, the lower breadth $b = \frac{25}{3,609} -$ $3,609\ cotang.\ 40^\circ = 6,927 - 4,301 = 2,626$ feet the slope or cut of the banks $= 3,609\ .\ cotang.\ 40^\circ = 4,301$, the upper breadth $= 6,927 + 4,301 = 11,228$ feet, the perimeter $p = b + \frac{2a}{sin.\ \Theta} = 2,626 + \frac{7,218}{sin.\ 40^\circ} = 13,855$ feet, and the ratio determining the friction $\frac{p}{F} = \frac{13,855}{25} = 0,5542$.

§ 366. The dimensions of the most suitable profiles which correspond to different angles of slope and to a given profile are to be found in the following table.

Angle of slope. Θ	Relative slope.	Dimensions of transverse profile. Depth a.	Lower breadth b.	Absolute slope na.	Upper breadth $b + 2na$.	Quotient $\frac{p}{F}$
90°	0	$0{,}707 \sqrt{F}$	$1{,}414 \sqrt{F}$	0	$1{,}414 \sqrt{F}$	$\frac{2{,}828}{\sqrt{F}}$
60°	0,577	$0{,}760 \sqrt{F}$	$0{,}877 \sqrt{F}$	$0{,}439 \sqrt{F}$	$1{,}755 \sqrt{F}$	$\frac{2{,}632}{\sqrt{F}}$
45°	1,000	$0{,}740 \sqrt{F}$	$0{,}613 \sqrt{F}$	$0{,}740 \sqrt{F}$	$2{,}092 \sqrt{F}$	$\frac{2{,}704}{\sqrt{F}}$
40°	1,192	$0{,}722 \sqrt{F}$	$0{,}525 \sqrt{F}$	$0{,}860 \sqrt{F}$	$2{,}246 \sqrt{F}$	$\frac{2{,}771}{\sqrt{F}}$
36° 52′	1,333	$0{,}707 \sqrt{F}$	$0{,}471 \sqrt{F}$	$0{,}943 \sqrt{F}$	$2{,}357 \sqrt{F}$	$\frac{2{,}828}{\sqrt{F}}$
35°	1,402	$0{,}697 \sqrt{F}$	$0{,}439 \sqrt{F}$	$0{,}995 \sqrt{F}$	$2{,}430 \sqrt{F}$	$\frac{2{,}870}{\sqrt{F}}$
30°	1,732	$0{,}664 \sqrt{F}$	$0{,}356 \sqrt{F}$	$1{,}150 \sqrt{F}$	$2{,}656 \sqrt{F}$	$\frac{3{,}012}{\sqrt{F}}$
26° 34′	2,000	$0{,}636 \sqrt{F}$	$0{,}300 \sqrt{F}$	$1{,}272 \sqrt{F}$	$2{,}844 \sqrt{F}$	$\frac{3{,}144}{\sqrt{F}}$
Semicircle		$0{,}798 \sqrt{F}$			$1{,}596 \sqrt{F}$	$\frac{2{,}507}{\sqrt{F}}$

We see from this table that the quotient $\frac{p}{F}$ is least for the semicircle, namely, $= \frac{2{,}507}{\sqrt{F}}$; greater for the semi-hexagon, and greater still for the half square, and the trapezium of 36° 52′, &c.

Example. What dimensions must be given to a profile, which has for an area of 40 square feet, a slope of its banks of 35°? From the preceding table, the depth $a = 0{,}697 \sqrt{40} = 4{,}408$, the lower breadth $= 0{,}439 \sqrt{40} = 2{,}777$ feet, the absolute slope $= 0{,}995 \sqrt{40} = 6{,}293$ feet, the upper breadth $= 15{,}363$, and the quotient

$$\frac{p}{F} = \frac{2{,}870}{\sqrt{40}} = 0{,}4538.$$

§ 367. *Uniform Motion.*—The motion of water in beds is for a certain tract either *uniform* or *variable;* it is uniform when the mean velocity at all transverse sections of this length remains the same, and therefore, also, the areas of the sections equal; and variable, on the other hand, when the mean velocities, and therefore, also, the areas of the sections vary. We shall treat first of uniform motion.

In the uniform motion of water along the distance $AD = l$, Fig. 489, the whole fall $HD = h$ is expended in overcoming the friction of the water in the bed, because the water flows on with the same velocity with which it arrives, therefore, a height due to a velocity is neither taken up nor set free. If we measure this friction by the height of this column of water, we may then make the fall equal to this height. But the height due to the resistance of friction increases

with the quotient $\frac{p}{F}$, with l and with the square of the mean velocity c (§ 329); hence then the formula holds good:

$$1.\ h = \zeta \cdot \frac{lp}{F} \cdot \frac{c^2}{2g},$$

in which ζ expresses a number deduced from experiment which may be called the *co-efficient of the resistance of friction.*

By inversion it follows:

$$2.\ c = \sqrt{\frac{F}{\zeta \cdot lp} \cdot 2gh.}$$

In determining the fall, therefore, when the length, the cross section and the velocity are given, and inversely, in deducing the velocity from the fall, the length and the cross section, we must know the co-efficient of friction ζ. According to Eytelwein's reduction of the ninety-one observations of Du Buat, Brünings, Funk and Waltmann, $\zeta = 0{,}007565$, and hence

$$h = 0{,}007565 \cdot \frac{lp}{F} \cdot \frac{c^2}{2g}.$$

If we put $g = 9{,}809$ metres or 31,25 feet (32,2 feet English), we have for the metrical measure

$$h = 0{,}0003856\, \frac{lp}{F} \cdot c^2 \text{ and } c = 5{,}09 \sqrt{\frac{Fh}{pl}},$$

and for the foot measure:

$$h = 0{,}00011726\, \frac{lp}{F} \cdot c^2 \text{ and } c = 92{,}35 \sqrt{\frac{Fh}{pl}} \text{ English measure.}$$

For conduit pipes $\frac{lp}{F} = \frac{\pi\, l\, d}{\frac{1}{4}\pi d^2} = \frac{4l}{d}$, hence this formula gives for pipes $h = 0{,}03026\, \frac{l}{d} \cdot \frac{v^2}{2g}$, whilst we have found more correctly for these (§ 331) for mean velocities

$$h = 0{,}025\, \frac{l}{d} \cdot \frac{v^2}{2g}.$$

The friction, therefore, as might be expected, is greater in the beds of rivers than in metallic conducting pipes.

Examples. 1. What fall must be given to a canal of the length $l = 2600$ feet, lower breadth $b = 3$ feet, upper breadth $b_1 = 7$ feet, and depth $a = 3$ feet, if it is to conduct a quantity of water of 40 cubic feet per second? It is:

$p = 3 + 2\sqrt{2^2 + 3^2} = 10{,}211$, $F = \frac{(7+3)\,3}{2} = 15$ and $c = \frac{40}{15} = \frac{8}{3}$, hence the fall sought, $h = 0{,}0001173 \cdot \frac{2600 \cdot 10{,}211}{15} \cdot (\frac{8}{3})^2 = \frac{0{,}305 \cdot 10{,}211 \cdot 64}{15 \cdot 9} = 1{,}476$ feet.—2. What quantity of water does a canal 5800 feet long, having a 3 feet fall, 5 feet deep, 4 feet lower and 12 feet upper breadth? Here:

$$\frac{p}{F} = \frac{4 + 2\sqrt{5^2 + 4^2}}{5 \cdot 8} = \frac{16{,}806}{40} = 0{,}42015;$$

hence the velocity

$$c = 92{,}35 \sqrt{\frac{3}{0{,}42015 \cdot 5800}} = \frac{92{,}35}{\sqrt{0{,}14005 \cdot 5800}} = \frac{92{,}35}{\sqrt{812{,}29}} = \frac{92{,}35}{28{,}5} = 3{,}24 \text{ feet,}$$

and the quantity of water $Q = Fc = 40 \cdot 3{,}24 = 129{,}6$ cubic feet, English measure.

§ 368. *Co-efficients of Friction.*—The co-efficient of friction for rivers, brooks, &c., the mean value of which, in the foregoing paragraphs, we have taken at 0,007565, is not constant, but, as in pipes, increases somewhat for small and diminishes for great velocities. We have, therefore, to put:

$$\zeta = \zeta_1 \left(1 + \frac{a}{c}\right) \text{ or } \zeta_1 \left(1 + \frac{a}{\sqrt{c}}\right).$$

The author of the work alluded to in § 363, finds from 255 experiments, the greater part of them undertaken by himself, for the Prussian measure $\zeta = 0{,}007409 \left(1 + \frac{0{,}0299}{c}\right)$, and hence it follows for the metre $\zeta = 0{,}007409 \left(1 + \frac{0{,}00939}{c}\right)$,

and for English measure $007409 \left(1 + \frac{0{,}0308}{c}\right)$.

It is manifest that these formulæ, for a velocity $c = 1\frac{1}{2}$ feet, give again the above assigned mean co-efficient of resistance $\zeta = 0{,}007565$. The following useful table of the co-efficients of resistance in the metrical measure serves for facilitating calculation.

Velocity c.	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	Meter.
Co-efficient of resistance $\zeta = 0{,}00$	811	776	764	758	755	753	751	750	749	

Velocity c.	1	1,2	1,5	2	3	Meter.
Co-efficient of resistance $\zeta = 0{,}00$	748	747	746	744	743	

The following table serves for the Prussian or English measure:

Velocity c.	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1	$1\frac{1}{2}$	2	3	5	10 ft.
Co-efficient of resistance $\zeta = 0{,}00$	815	797	785	778	773	769	766	763	759	752	749	745	743

These tables find their direct application in all cases where the velocity c is given and the fall to be found, and where the formula No. 1 of the former paragraph is applicable. But if the velocity c is unknown, and its amount to be determined, these tables will then only admit of a direct application, when we have already an approximate value of c. We may set to work in the simplest manner by determining c approximately by the formula $c = 50{,}9 \sqrt{\frac{Fh}{pl}}$, and from

this a value of ζ, taken from the table, and the value so obtained put into the formula

$$\frac{c^2}{2g} = \frac{h}{\zeta} \cdot \frac{F}{lp} \text{ or } c = \sqrt{\frac{F}{\zeta\, l p} \cdot 2\, gh}.$$

From the velocity c, the quantity of water is then given by the formula $Q = Fc$.

If, lastly, the quantity and the fall are given, and, as is often requisite in the construction of canals, it be required to determine the transverse section, we may put $\frac{p}{F} = \frac{m}{\sqrt{F}}$ (*see* Table, § 366) and $c = \frac{Q}{F}$ into the formula $h = 0{,}007565\, \frac{lp}{F} \cdot \frac{c^2}{2g}$, and write, therefore, $h = 0{,}007565\, \frac{m\, l\, Q^2}{2gF^{\frac{5}{2}}}$, and accordingly determine:

$F = \left(0{,}007565 \frac{m\ l\ Q^2}{2\ gh}\right)^{\frac{2}{5}}$, *i. e.*, for the metre $F = 0{,}0431\ \left(\frac{m\ l\ Q^2}{h}\right)^{\frac{2}{5}}$ or the English foot measure $F = 0{,}0268 \left(\frac{m\ l\ Q^2}{h}\right)^{\frac{2}{5}}$. Hence it follows, approximately, that $c = \frac{Q}{F}$; if we take a correspondent value of ζ from one of the tables, more accurately $F = \left(\zeta \cdot \frac{m\ l\ Q^2}{2\ gh}\right)^{\frac{2}{5}}$; and hence, more exact values for $c = \frac{Q}{F}$, $p = m\sqrt{F}$, as also for a, b, &c.

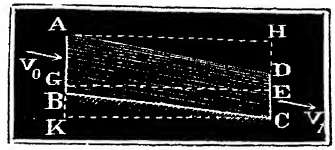
Examples.—1. What fall does a canal 1500 feet long, 2 feet lower and 8 feet upper breadth, and 4 feet depth require to give a discharge of 70 cubic feet per second? It is $p = 2 + 2\sqrt{4^2 + 3^2} = 12$, $F = 5 \cdot 4 = 20$, $c = \frac{70}{20} = 3{,}5$, hence $\zeta = 0{,}00748$, and $h = 0{,}00748 \cdot \frac{1500 \cdot 12}{20} \cdot \frac{3{,}5^2}{2g} = 6{,}732 \cdot 0{,}1902 = 1{,}28$ ft. (Eng.)—2. What discharge does a brook 40 feet broad, $4\frac{1}{2}$ feet mean depth, and 46 feet water profile, if it has a fall of 10 inches for a length of 750 feet? It is about $c = 92{,}35 \cdot \sqrt{\frac{40 \cdot 4{,}5 \cdot 10}{46 \cdot 750 \cdot 12}} = \frac{92{,}35}{\sqrt{230}} = 6{,}089$ feet, and hence $\zeta = 0{,}00745$. Hence we obtain, more correctly: $\frac{c^2}{2g} = \frac{Fh}{\zeta lp} = \frac{4{,}5 \cdot 40 \cdot 10}{0{,}00745 \cdot 46 \cdot 750 \cdot 12} = \frac{1}{1{,}7112} = 0{,}5844$, and $c = 6{,}119$ feet. Lastly, the corresponding discharge is $Q = 4{,}5 \cdot 40 \cdot 6{,}119 = 1101$ cubic feet, (Eng.)—3. A trench 3650 feet long is to be cut, which for a total fall of 1 foot is to carry off a discharge of 12 cubic feet per second, what dimensions are to be given to the transverse profile, if it is to preserve a regular semi-hexagonal figure? Here $m = 2{,}632$ (*see* Table, § 366), hence, approximately, $F = 0{,}0268\ (2{,}632\ \cdot 3650 \cdot 144)^{\frac{2}{5}} = 7{,}665$ square feet, and $c = \frac{12}{7{,}665} = 1{,}539$ feet. Hence ζ is to be taken $= 0{,}00758$, and $F = \left(0{,}00758 \cdot 2{,}632 \cdot \frac{3650 \cdot 144}{6{,}44}\right)^{\frac{2}{5}} = 7{,}67$ square feet. Therefore the depth must be made: $a = 0{,}760\sqrt{F} = 2{,}104$ feet, the lower breadth $= 0{,}877\sqrt{F} = 2{,}428$, and the upper breadth $= 2 \cdot 2{,}428 = 4{,}846$ English feet.

§ 369. *Variable Motion.*—The theory of the variable motion of water in beds of rivers may be reduced to the theory of uniform mo-

tion, provided the resistance of friction for a short length of the river may be considered as constant, and the corresponding height, in like manner, as $= \zeta \cdot \frac{lp}{F} \cdot \frac{v^2}{2g}$. But, besides this, regard must be had to the *vis viva* of the water, which corresponds to a change of velocity.

Let $ABCD$, Fig. 499, be a short extent of river, of the length $AD = l$, the fall $DH = h$, and let v_0 be the velocity of the arriving, and v_1 that of the departing water. If we apply the rules of efflux to an element D of the surface, we shall obtain for its velocity v_1,

Fig. 499.



$$\frac{v_1^2}{2g} = h + \frac{v_0^2}{2g};$$

as regards an element E below the surface, it is true that on the one side it has a greater height of pressure $AG = EH$; but as the downstream water reacts with a pressure DE, there remains for it only the fall $DH = EH - ED$, as pressure inducing motion, and so, for this or any other element, the formula:

$$h = \frac{v_1^2 - v_0^2}{2g} \text{ answers};$$

and if, further, the resistance due to friction be added, we then obtain:

$$h = \frac{v_1^2 - v_0^2}{2g} + \zeta \cdot \frac{lp}{F} \cdot \frac{v^2}{2g},$$

where p, F and v are the mean values of the wetted perimeter, transverse section, and velocity. If F_0 is the area of the upper, and F_1 that of the lower section, we may then put:

$$F = \frac{F_0 + F_1}{2}, \text{ and } Q = F_0 v_0 = F_1 v_1,$$

whence it follows that:

$$\frac{v_1^2 - v_0^2}{2g} = \frac{1}{2g}\left[\left(\frac{Q}{F_1}\right)^2 - \left(\frac{Q}{F_0}\right)^2\right] = \left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g}, \text{ and}$$

$$\frac{v^2}{F} = \frac{v_0^2 + v_1^2}{F_0 + F_1} = \left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{F_0 + F_1}, \text{ we obtain:}$$

$$1.\ h = \left[\frac{1}{F_1^2} - \frac{1}{F_0^2} + \zeta \frac{l\,p}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\right]\frac{Q^2}{2g}, \text{ as also}$$

$$2.\ Q = \frac{\sqrt{2gh}}{\sqrt{\frac{1}{F_1^2} - \frac{1}{F_0^2} + \zeta \frac{lp}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)}}.$$

The corresponding fall h may be calculated by means of the formula 1, from the quantity of water, the length and transverse section of a river or canal; and, inversely, the quantity of water from the fall, the length and the transverse section, by formula 2. To obtain greater accuracy, we may make the calculation for several short portions of the river, and take the arithmetical mean. If the total fall

only is known, we must substitute this at once for h in the last formula, and put

$$\frac{1}{F_1^2} - \frac{1}{F_0^2} = \frac{1}{F_n^{\ 2}} - \frac{1}{F_0^2},$$

where F_n denotes the area of the last section, and in place of

$$\zeta \cdot \frac{l\,p}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)$$

the sum of all similar values of the separate lengths of the river.

Example. A brook has for a distance of 300 feet a fall of 9,6 inches, the mean perimeter of its water profile is 40 feet, the area of the upper transverse profile 70, that of the lower 60 square feet; what quantity of water does this brook discharge? It is

$$Q = \frac{8{,}02\sqrt{0{,}8}}{\sqrt{\frac{1}{60^2} - \frac{1}{70^2} + 0{,}007567 \cdot \frac{300\,.\,40}{130}\left(\frac{1}{60^2} + \frac{1}{70^2}\right)}} =$$

$$= \frac{7{,}173}{\sqrt{0{,}0000731 + 0{,}0003365}} = \frac{7{,}173}{\sqrt{0{,}0004096}} = 354{,}43 \text{ cubic feet.}$$ The mean velocity is $\frac{2\,Q}{F_0 + F} = \frac{708{,}8}{130} = 5{,}452$ feet; hence, more accurately, ζ must be taken $= 0{,}00745$ in place of $0{,}007565$, and therefore more nearly:

$$Q = \frac{7{,}173}{\sqrt{0{,}0000731 + 0{,}0003314}} = 357{,}5 \text{ cubic feet.}$$ If the same brook, with the same head of water, had for a length of 450 feet, a fall of 11 inches, and if its upper transverse profile had an area of 50 and its lower of 60 square feet, and the mean perimeter of the profile measured 36 feet, we should then have:

$$Q = \frac{8{,}02\sqrt{0{,}9167}}{\sqrt{\frac{1}{60^2} - \frac{1}{50^2} + 0{,}00745 \cdot \frac{450\,.\,36}{110}\left(\frac{1}{60^2} + \frac{1}{50^2}\right)}}$$

$$= 8{,}02\sqrt{\frac{0{,}9167}{0{,}0001222 + 0{,}0007436}} = 308 \text{ cubic feet.}$$

The mean of these two values is $Q = \frac{357\,.\,5 + 308}{2} = 332{,}75$ cubic feet.

§ 370. In order to obtain a formula for the depth of water, let the upper depth $= a_0$ and the lower $= a_1$, the slope of the bed $= \alpha$, consequently the fall of the bed $= l\ sin.\ \alpha$. We then obtain the fall of the water $h = a_0 - a_1 + l\ sin.\ \alpha$, and there results the equation:

$$a_0 - a_1\left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g} = \left[\zeta\,\frac{p}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{2g} - sin.\ \alpha\right] l,$$

$$\text{hence } l = \frac{a_0 - a_1 - \left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g}}{\zeta\,\frac{p}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{2g} - sin.\ \alpha}.$$

The length l which corresponds to a difference $a_0 - a_1$ of the depth of water, may be determined by this formula. But if the reverse problem is to be solved, we must do it by the method of approximation, and first determine the distances l_1 and l_2 corresponding to the assumed depressions $a_0 - a_1$, and $a_1 - a_2$, and from these calculate by a proportion, the depression corresponding to a given distance l.*

* See "Ingenieur," Arithmetik, § 16, v.

The formula is further capable of simplification when the breadth b of the running water is constant, or may be considered as such. In this case we put:

$$\left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g} = \frac{F_0^2 - F_1^2}{F_0^2 F_1^2} \cdot \frac{Q^2}{2g} = \frac{(F_0 - F_1)(F_0 + F_1)}{F_1^2} \cdot \frac{v_0^2}{2g}$$

$$= \frac{(a_0 - a_1)(a_0 + a_1)}{a_1^2} \cdot \frac{v_0^2}{2g} \text{ approximately } = 2\,\frac{(a_0 - a_1)}{a_0} \cdot \frac{v_0^2}{2g},$$

and likewise:

$$\frac{p}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{2g} = \frac{p\,(F_0^2 + F_1^2)}{(F_0 + F_1)F_1^2} \cdot \frac{v_0^2}{2g} \text{ approximately}$$

$$= \frac{p}{a_0 b} \cdot \frac{v_0^2}{2g}, \text{ hence } l = \frac{(a_0 - a_1)\left(1 - \frac{2}{a^0} \cdot \frac{v_0}{2g}\right)}{\zeta \cdot \frac{p}{a_0 b} \cdot \frac{v_0^2}{2g} - sin.\ \alpha}, \text{ and hence}$$

$$\frac{a_0 - a_1}{l} = \frac{\zeta \cdot \frac{p}{a_0 b} \cdot \frac{v_0^2}{2g} - sin.\ \alpha.}{1 - \frac{2}{a_0} \cdot \frac{v_0^2}{2g}}.$$

The difference $(a_0 - a_1)$ of the depth corresponding to a given extent l may be calculated directly by this formula.

Example. In a horizontal trench, 5 feet broad and 800 feet long, it is desired to carry off a 20 cubic feet discharge, and to let it flow in at a depth of 2 feet, what depth will the water at the end of the canal have? Let us divide the whole length into two equal portions, and determine from the last formula the fall for each of them.

Here the $sin.\ \alpha = 0$, $l = \frac{800}{2} = 400$, and $b = 5$; for the first portion $v = \frac{20}{2\,.\,5} = 2$, hence $\zeta = 0{,}00752$, also $a_0 = 2$; since $p = 8\frac{1}{2}$, it follows that $a_0 - a_1 = \left(\frac{0{,}00752 \,.\, \frac{8{,}5}{10} \cdot \frac{4}{2g}}{1 - \frac{2}{2} \cdot \frac{4}{2g}}\right) . 400 = \frac{0{,}15877}{0{,}9379} = 0{,}1692$ feet. Now, for the second half, $a_1 = 2 - 0{,}1692 = 1{,}8308$, and $p_1 = 8{,}2$, $v_1 = \frac{20}{9{,}154} = 2{,}1848$, and the depression of the second portion:

$$a_1 - a_2 = \left(\frac{0{,}00752 \,.\, \frac{8{,}2}{9{,}154} \cdot \frac{2{,}1848^2}{2g}}{1 - \frac{2}{1{,}8308} \cdot \frac{2{,}1848^2}{2g}}\right) . 400 = \frac{0{,}1997}{0{,}919} = 0{,}2173 \text{ feet, hence the}$$

whole depression $= 0{,}1692 + 0{,}2173 = 0{,}3865$, and the depth of water at the lower end $= 2 - 0{,}3865 = 1{,}6135$.

§ 371. *Floods.*—When the depth of water in rivers and canals varies, variations in the velocity and discharge take place likewise. A greater depth of water not only involves a greater section, but also a greater velocity, and hence, for two reasons, a greater quantity of water, and likewise a diminution of the depth of water, gives a diminution of the section and the velocity, and hence also a decrease of the discharge. If the original depth $= a$, and any increased depth

$= a_1$, the upper breadth of the canal $= b$, then the augmentation of the section may be put $= b\,(a_1 - a)$, and hence the section afterwards $a_1 - a$, $F_1 = F + b\,(a_1 - a)$, it also follows from this that

$$\frac{F_1}{F} = 1 + \frac{b\,(a_1 - a)}{F}, \text{ and}$$

$$\sqrt{\frac{F_1}{F}} \text{ approximately} = 1 + \frac{b\,(a_1 - a)}{2\,F}.$$

If further p be the original, p_1 the increased perimeter of the water profile, and Θ the angle of slope of the banks, then

$$p_1 = p + \frac{2\,(a_1 - a)}{p\, sin.\, \Theta}, \text{ hence } \frac{p_1}{p} = 1 + \frac{2\,(a_1 - a)}{p\, sin.\, \Theta}, \text{ and}$$

$$\sqrt{\frac{p_1}{p}} = 1 + \frac{a_1 - a}{sin.\, \Theta}, \text{ as also } \sqrt{\frac{p}{p_1}} = 1 - \frac{a_1 - a}{p\, sin.\, \Theta}.$$

Now the velocity with the first depth of water is

$$c = 92{,}35 \sqrt{\frac{Fh}{pl}}, \text{ and with the second } c_1 = 92{,}35 \sqrt{\frac{F_1}{p_1} \cdot \frac{h}{l}},$$

hence we may put:

$$\frac{c_1}{c} = \sqrt{\frac{F_1}{F}} \cdot \sqrt{\frac{p}{p_1}} = \left(1 + \frac{b\,(a_1 - a)}{2 \cdot F}\right)\left(1 - \frac{a_1 - a}{p\, sin.\, \Theta}\right)$$

$$= 1 + (a_1 - a)\left(\frac{b}{2F} - \frac{1}{p\, sin.\, \Theta}\right),$$

therefore the relative change of velocity:

$$1.\quad \frac{c_1 - c}{c} = (a_1 - a)\left(\frac{b}{2\,F} - \frac{1}{p\, sin.\, \Theta}\right).$$

On the other hand, the ratio of the discharge is:

$$\frac{Q_1}{Q} = \frac{F_1 c_1}{Fc} = \left(1 + \frac{b\,(a_1 - a)}{F}\right)\left[1 + (a_1 - a)\left(\frac{b}{2\,F} - \frac{1}{p\, sin.\, \Theta}\right)\right]$$

$$= 1 + (a_1 - a)\left(\frac{3\,b}{2F} - \frac{1}{p\, sin.\, \Theta}\right),$$

and the relative increase:

$$2.\quad \frac{Q_1 - Q}{Q} = (a_1 - a)\left(\frac{3\,b}{2F} - \frac{1}{p\, sin.\, \Theta}\right).$$

Less accurately, but in many cases, especially in broad canals with little slope, we may put $F = ab$, and neglect $\frac{1}{p\, sin.\, \Theta}$, whence it follows more simply that:

$$\frac{c_1 - c}{c} = \tfrac{1}{2}\,\frac{a_1 - a}{a}, \text{ and}$$

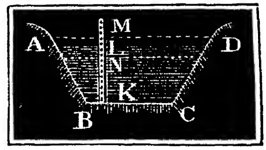
$$\frac{Q_1 - Q}{Q} = \tfrac{3}{2} \cdot \frac{a_1 - a}{a}.$$

From this, therefore, the relative change of velocity is $\tfrac{1}{2}$, *and the relative change in the quantity of water* $\tfrac{3}{2}$, *that of the relative change in the depth of water.*

Examples.—1. When the head of water increases $\tfrac{1}{10}$ of its original amount, the velo-

city is then $\frac{1}{20}$, and the quantity $\frac{3}{20}$ greater than its original value.—2.—When the depth diminishes 8 per cent., the velocity then diminishes 4, and the quantity 12 per cent.—3. From the more correct formula:

Fig. 500.



$$\frac{Q_1 - Q}{Q} = (a_1 - a)\left(\frac{3\,b}{2F} - \frac{1}{p\ sin.\ \Theta}\right)$$

a scale of the depth of water KM, Fig. 500, may be constructed, on which the discharge of a canal corresponding to any depth KL, may be read off, when the quantity of water for a certain mean depth is once known. If $b = 9$ feet, $b_1 = 3$, $a = 3$, and $\Theta = 45°$, we then have $F = \frac{(9 + 3)\,3}{2} = 18$ square ft., $p = 3 + 2\,.\,3\sqrt{2} = 11{,}485$ and $sin.\ \Theta = \sqrt{\frac{1}{2}} = 0{,}707$, hence:

$$\frac{Q_1 - Q}{Q} = \left(\frac{3\,.\,9}{2\,.\,18} - \frac{1}{11{,}485\,.\,0{,}707}\right)(a_1 - a) = (0{,}750 - 0{,}123)\,(a_1 - a) = 0{,}627\,(a_1 - a).$$

If the quantity corresponding to a mean head of water $Q = 40$ cubic feet, we then have $Q_1 = 40 + 40\,.\,0{,}627\,(a_1 - a) = 40 + \frac{a_1 - a}{0{,}04}$. If $a_1 - a = 0{,}04$ feet $= 5{,}76$ lines, it follows that $Q_1 = 41$; $a_1 - a = 0{,}08$ feet $= 11{,}52$ lines, we then have $Q_1 = 42$ cubic feet; if, further, $a_1 - a = -0{,}04$, then is $Q_1 = 39$ cubic feet, &c. A scale, therefore, whose intervals are $LM = LN = 5{,}76$ lines, gives the discharge accurately to a cubic foot. Of course the accuracy is the less, the more the head of water differs from a mean value.

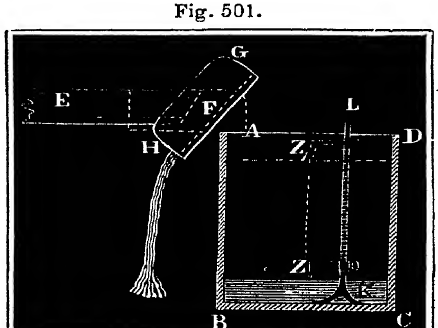
Remark. The conducting and carrying off of water in canals, as well as the subject of weirs and dams, will be fully treated of in the Second Part.

CHAPTER VIII.

HYDROMETRY, OR THE DOCTRINE OF THE MEASUREMENT OF WATER.

§ 372. *Gauges.*—The quantity, which a stream discharges in a certain time, is determined either by a gauge, by an apparatus of efflux, or by an hydrometer. The most simple way of measuring water is by the guage, *i. e.* by the use of a graduated vessel, but this method is only applicable to small discharges, carried off by pipes or small brooks, or drains. The gauge vessel is generally made of wood, and of a rectangular form, and to increase its strength is bound round with iron-hooping. The water is conducted into it by a trough EF, Fig. 501, at whose extremity there is a double valve GH, by which the water may be made to flow at will into the vessel AC, or by the side of it. To obtain the exact depth of the body of water in the vessel, a scale KL is fur-

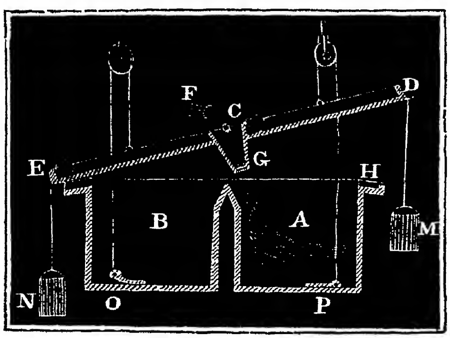
Fig. 501.



ther applied. If before measurement, the index Z be moved down to the surface of the water, already in the vessel, and merely covering the bottom, and the head of water read off from the scale, we shall obtain the height ZZ_1 of the gauged water by subtraction of this from the head of water which the scale indicates when the index hand Z_1 is brought into contact with the surface of water at the end of the observation. Before measurement, the valve must be so placed that the water may flow off outside the vessel. When we are convinced that the efflux in the trough is in a state of permanency, and, watch in hand, have noted a certain moment, the valve must then be turned, so that the water may run into the gauge vessel, and after it is either partly or entirely filled, a second interval is noted by the watch, and the valve again brought into its first position. From the mean section F of the vessel, and the depth $ZZ_1 = a$ of the body of water, the whole quantity $= Fa$ may be estimated, and again from the time of filling t, given by the difference of the times observed, the quantity of water per second $Q = \frac{Fa}{t}$.

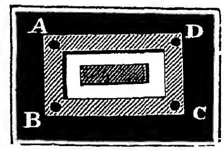
Remark. To determine a variable quantity of efflux at each period of the day, we may make use of the apparatus represented in Fig. 502, as applicable especially in salt works. There are here two gauge vessels, A and B, which alternately fill and empty themselves, and the water which is conducted by the pipe F passes through a short pipe CG, which is rigidly connected with a lever DE, revolving about C. When one vessel A becomes filled, the water then flows through a short tube H into the little vessel M, this draws the lever down again on one side, and the pipe CG comes into such a position that the water is conducted into B. The drawing up of the valves O and P takes place by means of strings passing over pulleys, whose extremities are connected with the lever, and sustained by iron balls, which impart a final impulse to the descent of the lever. The vessels M and N have small efflux orifices, by which they empty themselves after each reversion of the lever. An apparatus is besides applied, by which the number of strokes may be read off at any time.

Fig. 502.



§ 373. *Efflux Regulators.*—Small and medium discharges are very frequently determined by means of their flow through a definite orifice, and under a known head. From the area F of the orifice, the head of water h, and the efflux co-efficient μ, the discharge per second $Q = \mu F \sqrt{2gh}$ is given. The Poncelet orifices are those best adapted for this purpose, because the co-efficients of efflux of these under different heads of water are known with great accuracy (§ 316), still they are applicable only to certain medium discharges. The author availed himself of four such orifices for his measurements, one of five, one of ten, one of fifteen, one of twenty centimetres depth, but all of twenty centimetres width. These orifices are cut out of brass plate,

Fig. 503.



and fixed to a wooden frame AC, Fig. 503, which is fastened by four strong iron screws to each wall. In many cases, indeed, greater orifices, the co-efficients of efflux for which are not so accurately determined, and sometimes wiers must be used, which admit generally of a still less accuracy. In all cases, however, the rule holds good, that we must endeavor to get as complete and perfect a contraction as possible, and for this reason must give to the orifice, if it is in a thick plate, a slope outward. The corrections which must be applied for incomplete and partial contraction, have been sufficiently distinguished in paragraphs 319, 320, &c. To measure the water of a brook we must set the frame with its orifice, and wait for the moment when the head of water is permanent. For the measurement of the head of water we must avail ourselves of the index scale, Fig. 500, or of the movable scale EF, Fig. 505. If we would note the efflux directly from the apertures of sluices, it is better to fix before hand a pair of brass scales BC and DE, Fig. 504, with

Fig. 504.

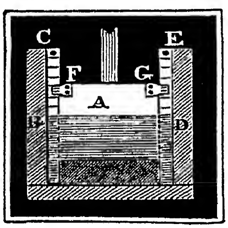
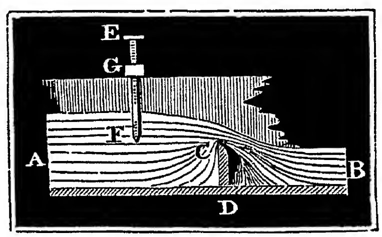


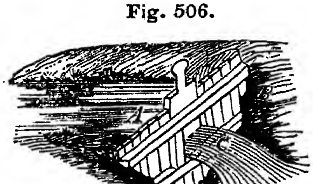
Fig. 505.



their indices F and G to the slide, and to the sluice-board A, in order to read off more safely the height of the aperture. It is generally better for the purpose of measuring water, to put on a new sluice-board with its guide, and with the requisite slope outwards. The simplest means of measuring water in a channel, consists in putting in a board CD, with its upper edge sloped off, Fig. 505, and measuring the fall produced by it. If the channel is long, and there is little rise, it is generally some time before the condition of permanency takes place, and it is for this reason good, before measurement, to put on a second board, so as to impede the efflux of water for a long time, in order to accelerate the rise to a height corresponding to a state of permanency.

To measure the quantity of water of a brook, we may dam it up with posts and boards as in Fig. 506, and let the water C run off through an aperture, or we may avail ourselves of a simple overfall or wier, but of this we shall treat in the second part.

Fig. 506.



§ 374. But as it is often long before a state of permanency occurs in water dammed up by this construction, we may adopt with

advantage the following method, first proposed by Prony. We may first close entirely the aperture by a sluice-board, and let the water rise to some height, or as high as circumstances will admit, then draw it so far up that more water may flow in than out, and measure the heads of water at equal and very short intervals; lastly, the aperture of the sluice must be again perfectly closed, and the time t in which the water rises to the first height, further noted. In each case, then, during the whole time of observation $t+t_1$, as much water flows in as out, and hence the quantity flowing in, in the time $t+t_1$, may be expressed by the quantity flowing out in the time t_1. If the heads of water during the depressions are h_0, h_1, h_2, h_3, and h_4, we have then the mean velocity of efflux:

$$v = \frac{\sqrt{2g}}{12} (\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}) \text{ (see § 351)},$$

and if the area of the aperture $= F$, we have then the quantity of efflux in the time t:

$$V = \frac{\mu F t \sqrt{2g}}{12} (\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}), \text{ and the}$$

quantity flowing in per second:

$$Q = \frac{V}{t+t_1} = \frac{\mu F t \sqrt{2g}}{12(t+t_1)} (\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}).$$

Example. To measure the water of a brook used for the driving of a water-wheel, which has been dammed up by a sluice, Fig. 506, after opening the rectangular aperture, the following is observed: the original head of water is 2 feet, after 30″ 1,8 feet, after 60″ 1,55 feet, after 90″ 1,3 feet, after 120″ 1,15 feet, after 150″ 1,05 feet, and after 180″ 0,9 feet, breadth of the aperture 2 feet, depth $\frac{1}{2}$ foot, time of rising to the first height with closed aperture $= 110''$. The mean velocity of efflux is:

$$v = \frac{8,02}{18} (\sqrt{2} + 4\sqrt{1,8} + 2\sqrt{1,55} + 4\sqrt{1,3} + 2\sqrt{1,15} + 4\sqrt{1,05} + \sqrt{0,9}) =$$

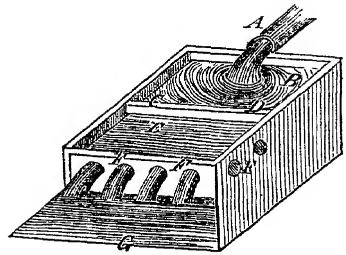
$0,440\,(1,414 + 5,364 + 2,490 + 4,561 + 2,145 + 4,099 + 0,949) = 0,440 \,.\, 21,022 =$ 9,248 feet; but now $F = 2 \,.\, \frac{1}{2} = 1$ square foot, hence it follows that the theoretical discharge is $= 9,248$ cubic feet. If the co-efficient of efflux is taken $= 0,61$, we finally obtain the quantity of water sought:

$$Q = \frac{0,61 \,.\, 180}{180 + 110} \,.\, 9,248 = 3,5015 \text{ cubic feet, (English.)}$$

§ 375. *The "Pouce d'Eau," or Water-Inch.*—To measure small discharges, we avail ourselves of the flow through round 1 inch wide orifices, in a thin plate, under a given pressure. The discharge given through such an aperture under the least pressure, or when the surface is only a line above the uppermost position of the orifice, is called an *inch of water.* The French assume for the water-inch (old Paris measure) 15 pints, or 19,1953 cubic metres of water in the 24 hours; therefore in 1 hour 0,7998, and in 1 minute 0,0133 cubic metres; yet older data, by Mariotte, Couplet, and Bossut, vary not a little from the above. According to Hagen, an inch of water (Prussian measure) delivers in 24 hours 520 cubic feet, therefore, in a minute, 0,3611 cubic feet. The double water modulus of Prony, which corresponds to an orifice of 2 centimetres diameter, with a pressure of 5 centi-

metres, and discharges 20 cubic metres of water in 24 hours, has not been adopted. The apparatus by which water is measured by the inch is represented in Fig. 507. The water to be measured flows through the tube A into a box; from this it passes through holes below in the partition CD into the box E, and from this through a horizontal row of round orifices F, of exactly 1 inch width, and cut in tin plate, into the reservoir G. That the surface of water may stand a line above the heights of these orifices, it is necessary that there be a sufficient number of them, and that a part of them be closed by stoppers. For great discharges the whole water is divided, and in this way a part, only one-tenth, is measured. This division may be accomplished easily, by first conducting the water into a reservoir, with a certain number of orifices at the same level, and only to receive the quantity delivered by one orifice in the apparatus represented above.

Fig. 507.



Remark 1. We may apply also cocks and other regulating apparatus to the measurement of water, if we know the co-efficient of resistance for each position. If h is the head of water, F the transverse section of the pipe, and μ the co-efficient of efflux, for a cock quite opened, we then have the discharge $Q = \mu\ F \sqrt{2\ g\ h}$, as inversely, $\mu = \frac{Q}{F\sqrt{2\ g\ h}}$ and $\frac{1}{\mu^2} = \left(\frac{F}{Q}\right)^2 .\ 2\ g\ h$. If now we put the co-efficient of resistance corresponding to a position of the cock, and taken from the tables already given $= \zeta$, we then have the corresponding discharge:

$$Q_1 = F\sqrt{\frac{2\ g\ h}{\frac{1}{\mu^2} + \zeta}} = \frac{\mu F \sqrt{2\ g\ h}}{\sqrt{1 + \mu^2 \zeta}} = \frac{Q}{\sqrt{1 + \zeta\left(\frac{Q}{F}\right)^2 \cdot \frac{1}{2\ g\ h}}}.$$

For convenience sake, we may construct for ourselves a table, so that we can find at a glance the discharge corresponding to a position of the cock, or the position of the cock corresponding to a given discharge. If, for example, $\mu = 0{,}7$ and $F = 5$ square inches, we have then:

$$Q_1 = \frac{0{,}7\ .\ 4\ .\ 12\ .\ 8{,}02 \sqrt{h}}{\sqrt{1 + 0{,}49\ .\ \zeta}} = 269{,}5 \sqrt{\frac{h}{1 + 0{,}49\ \zeta}} \text{ cubic inches,}$$

or if h is constantly 1 foot, $Q_1 = \frac{269{,}5}{\sqrt{1 + 0{,}49\ \zeta}}$. If now the positions of the cock are at 5°, 10°, 15°, 20°, 25°, &c., the co-efficients of resistance, 0,057; 0,293; 0,758; 1,559;

Fig. 508.

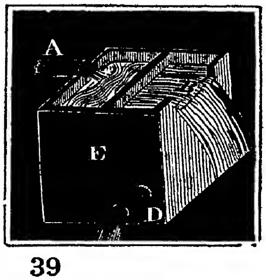
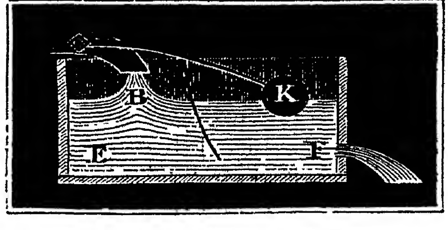


Fig. 509.



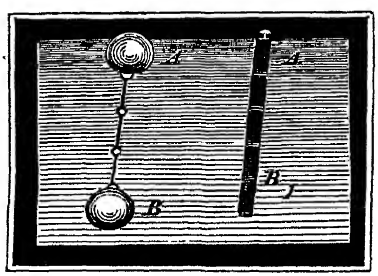
3,095, the discharges corresponding to these are: 265,8; 252; 230,1; 202,8; 169,9 cubic inches.

Remark 2. To regulate the flow through an orifice D, Fig. 508, we may apply a weir B that the excess of water from the pipe A may flow over, and that a constant pressure may be maintained in the reservoir DE. That there may be no loss of water, a cock or a valve A, Fig. 509, is applied, which is regulated by a float K acting upon a lever, so that as much water only flows in through B as flows out through F.

§ 376. *Floating Bodies.*—The discharge of large brooks, canals, and rivers, can be determined only by an hydrometer indicating the velocity. Of such instruments floating bodies are the most simple. We may use any floating body for this purpose, but it is better to have bodies of a moderate size, which are only a little specifically lighter than water. Substances of about $\frac{1}{10}$ of a cubic foot content are large enough. Very large ones do not easily assume the velocity of water, and very small ones again, especially when much above the water, are easily disturbed in their motion by accidental circumstances, sometimes by the air on the surface of the fluid. Often, plain pieces of wood are sufficient; it is better, however, if they have a coating of some bright varnish, and better still if the floats are hollow, such as glass flasks, tin balls, &c., because these may be filled at will with water. Swimming balls are most frequently used. They are from 4 to 12 inches diameter, and made of brass, and painted over with some light oil-color, to make them more visible to the eye, and have an opening with a neck, that they may be filled with water and stopped. A floating ball, such as A, Fig. 510, gives only the velocity at the surface, and often only that of the main current; but by suspending two balls one to the other, A and B, Fig. 511, we may determine the velocity at different depths. In this case, the one ball B, which swims under water, is quite filled with the fluid; the other, however, which swims on the surface, is only filled just enough to make it float a little above the surface. Both balls are connected with each other by a string or wire, or by a light wire chain. The velocity c_0 of the surface is first determined by the single ball, and then the mean velocity of the two observed by the connection of balls.

Fig. 510.

Fig. 511.



If, now, the velocity at the depth of the second ball be denoted by c_1, we may then put $c = \frac{c_0 + c_1}{2}$, and hence, inversely, $c_1 = 2\,c - c_0$. Whilst now both balls are connected with one another by longer and longer wires, we may, in this manner, find the velocities at greater depths. The mean velocity c of a perpendicular is otherwise given if the second ball is allowed to swim a little above the bottom, and c is made $= 2\,c_1 - c_0$; still more accurately, however, if for c_1 the mean of all the velocities observed in a perpendicular be taken.

To find the mean velocity in a perpendicular, the floating staff $A_1 B_1$, represented in Fig. 512, is used. This is particularly convenient for measurements in canals and cuts when it is composed of short pieces screwed together. The floating staff which the author uses is composed of 15 hollow portions, each 1 decimetre in length. That this may swim pretty nearly upright, the lowermost piece is loaded with shot, so that the top just rises above the water. The number of pieces composing the staff, depends, of course, on the depth of the canal.

Both with the floating staff as well as the connection of balls, it may be observed that the velocity at the surface, when the motion of the water in beds is unimpeded, is greater than at the bottom, because the top of the staff swims in advance of the bottom, and the upper ball in advance of the lower. In contraction only, for example, when the water is dammed up by piles, &c., does the contrary take place.

Remark. As a rule, especially with large and floating bodies, as ships, &c., the velocity of the swimming body is somewhat greater than that of the water; not so much because these bodies in swimming float down an inclined plane formed by the surface of the water, but because they take none, or scarcely any, part in the irregular intimate motion of the water; still, the variation for small floating bodies is so slight that it may be neglected.

§ 377. The velocity of a floating ball is found by noting the time t with a good seconds watch, or a half-second pendulum (§ 247), which it takes while floating on the water to describe a measured distance s, marked out on the banks. Then the required velocity of the ball is $c = \frac{s}{t}$. That the time t corresponding to the space described along the bank may be accurately found, it is necessary, with the assistance of a cross line or lines, to erect at the opposite bank two signal staves C and D, perpendicular at A and B, Fig. 512. If we place ourselves behind A, we may then note the moment when the float K, dropped in a little above A, comes into the line AC, and if behind B, we may then also observe the time by a watch held in the hand, when

Fig. 512.

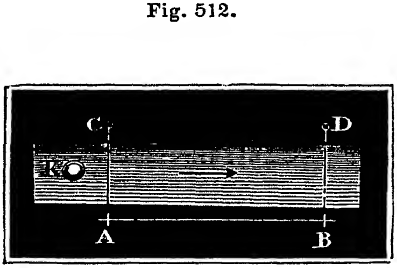
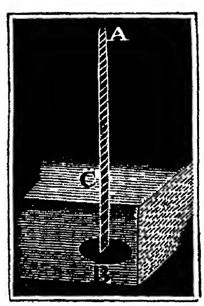


Fig. 513.



the float reaches the line BD, and we then find by subtraction of the times of observation, the required time t corresponding to the describing of the space s. Besides the mean velocity c of the water, the area

F of the transverse profile is further required for determining the quantity of water $Q = F c$. To find this area, it is necessary to know the breadth and the mean depth of the water. The depths are measured by a sounding rod AB, Fig. 513, having a rhomboidal section, and a board B at the foot; for greater depths we may also use a sounding chain, at whose extremity there is an iron plate, which, in sinking, rests on the bottom. The breadth and the abscissæ corresponding to the measured depths, or the distances from the banks in canals and small brooks EFG, Fig. 514, are found by stretching across a mea-

Fig. 514.

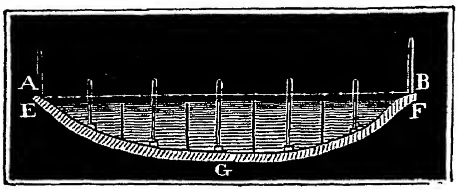
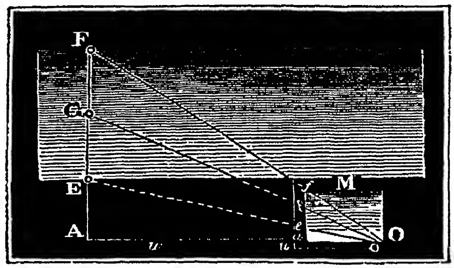


Fig. 515.



suring chain AB, or the placing of a rod right across the running water. For broad rivers this is determined by a measure table M, which is placed at a proper distance AO, from the section EF, Fig. 515, which is to be measured. If ao is the distance AO between A and O, reduced to the table, and if ao is placed in the direction of AO, and thereby also the direction of the breadth af made parallel to the line of breadth AF marked out, then each line of vision will intersect in the direction of the points E, F, G, &c., in the profile, the corresponding points e, f, g on the table, and ae, af, ag, &c., are the distances AE, AF, AG, &c., in the reduced measure. It is not, therefore, necessary on putting in the sounding rod, and measuring the depths by it, to measure the distances of the corresponding points of the banks, if the engineer standing by the measure table looks at the sound on its being put in, in the line EF.

If, now, the breadth EF, Fig. 514, of a transverse profile, consist of parts b_1, b_2, b_3, &c., and the mean depths within those parts a_1, a_2, a_3, and the mean velocities c_1, c_2, c_3, &c., we have then the area of the profile:

$$F = a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots,$$

the discharge:

$$Q = a_1 b_1 c_1 + a_2 b_2 c_2 + a_3 b_3 c_3 + \ldots,$$

and, lastly, the mean velocity:

$$c = \frac{Q}{F} = \frac{a_1 b_1 c_1 + a_2 b_2 c_2 + \ldots}{a_1 b_1 + a_2 b_2 + \ldots}.$$

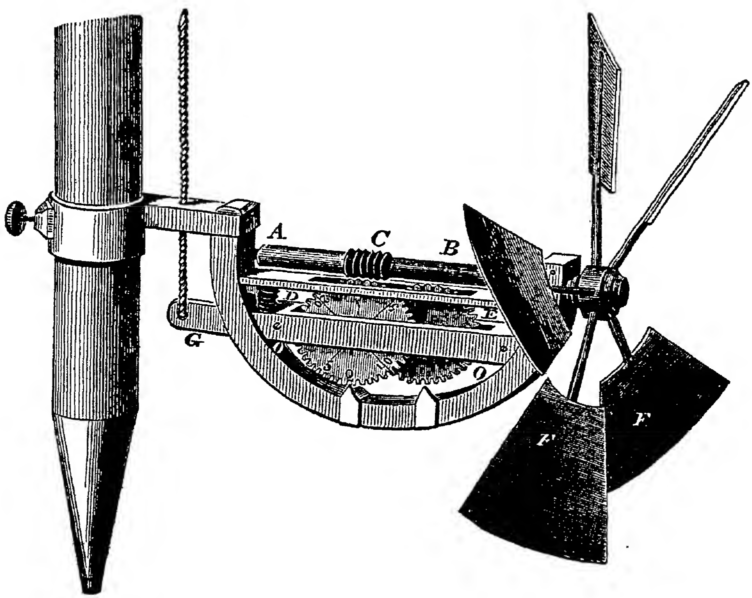
Example. In a tolerably straight and uniform extent of river, we have at the middle points of portions of the breadth:

	5 feet,	12 feet,	20 feet,	15 feet,	7 feet,
The depths - - - - -	3 "	6 "	11 "	8 "	4 "
The mean velocities - -	1,9 "	2,3 "	2,8 "	2,4 "	2,1

Hence we may put:
The area of the profile $F = 5 \,.\, 3 + 12 \,.\, 6 + 20 \,.\, 11 + 15 \,.\, 8 + 7 \,.\, 4 = 455$ square feet.
The quantity of water $Q = 15 \,.\, 1{,}9 + 72 \,.\, 2{,}3 + 220 \,.\, 2{,}8 + 120 \,.\, 2{,}4 + 28 \,.\, 2{,}1$ $= 1156{,}9$ cubic feet. The mean velocity $c = \frac{1156{,}9}{455} = 2{,}54$ feet.

§ 378. *The Tachometer.*—The most eligible hydrometer is the tachometer of Woltmann, Fig. 516. It consists of a horizontal axle AB, with from two to five vanes F, placed at an inclination to the direction of the axis, and gives, when immersed in the water and held at right angles to the direction of motion, by the number of its revolutions in a certain time, the velocity of the running water. To read off the number of these revolutions, the axle has a few turns of a screw C, and these work into the teeth of a wheel D, upon whose lateral surfaces numbers are engraved, which give, by means of an index, the number of revolutions of the wheel. But to be able to register a great number of revolutions upon the axle of this toothed

Fig. 516.



wheel, there is a pinion which works into the teeth of the wheel E, by which, like the hands of a watch, several multiple revolutions may be read off. If, for example, each of the two toothed wheels has fifty teeth, and the trundle ten, then the second wheel revolves one tooth whilst the first advances five teeth, or the vanes make five revolutions, if the index of the first wheel points to $27 = 25 + 2$, and that of the second to 32, the corresponding number of revolutions of the vanes is accordingly: $= 32 \,.\, 5 + 2 = 162$. The entire instrument is screwed to a staff having a tin vane attached, to admit of easy

immersion in the water, and of being kept opposed to the current. But that the wheelwork may only revolve during the time of observation, the axis is connected with a lever GO, which is pressed down by a spring, so that the teeth of the first wheel are thrown into gear with the screw only when the lever is drawn up by a string.

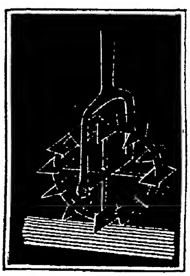
The number of revolutions of a wheel in a certain time, for example, in a second, is not exactly proportional to the velocity of the water, hence we cannot put $v = \alpha u$, where u is the number of revolutions, v the velocity, and α a number deduced from experiments; but rather: $v = v_0 + \alpha u$, or more correctly $v = v_0 + \alpha u + \beta u^2 \ldots$, or still more correctly:

$v = \alpha u + \sqrt{v_0^2 + \beta u^2}$, where v_0 is the velocity, at which the water is no longer able to turn the wheel, and α and β are co-efficients from experiment. The constants v_0, α and β, are to be determined for each instrument in particular. With their assistance the velocity is known from a single observation, nevertheless it is always safer to make at least two, and to substitute the mean value as the correct one.

Example. If for a sail-wheel $v_0 = 0{,}110$ feet, $\alpha = 0{,}480$, and $\beta = 0$, therefore $v = 0{,}11 + 0{,}48\, u$, and we have by an observation with this instrument found the number of revolutions 210 in 80'', then the corresponding velocity is:

$$v = 0{,}11 + 0{,}48 \cdot \frac{210}{80} = 0{,}11 + 1{,}26 = 1{,}37 \text{ feet.}$$

Fig. 517.



Remark 1. The constants v_0, α and β depend principally on the magnitude of the angle of impact, *i. e.*, on the angle which the plane of the vane makes with the direction of motion of the water, and therefore, also, with the direction of the axis of the wheel. To observe with tolerable accuracy small velocities, it is well to have a large angle of impulse, *i. e*, one of 70°. For the rest, it is desirable to have vanes of different sizes and with different angles of impulse, and to use the vane with small angles of impulse for great velocities, and a smaller one for shallow water.

Remark 2. To find the velocity of the surface of water, a small tin wheel may be used, as represented in Fig. 517, and its under part allowed to dip into the water. The number of its revolutions may be determined by a system of wheels, as in the tachometer.

§ 379. To find the constant or co-efficient of the tachometer, it is necessary to set this instrument in a stream, whose velocity is known, and to note the corresponding number of revolutions. Although as many observations only are required, as there are constants, it is still safer to have as many observations as possible, and especially for very different velocities, because we may then apply the method of least squares, and thereby eliminate the effect of accidental errors of observation. The velocity of the water may be found by the floating ball, or by receiving the water in a gauge vessel, and dividing the measured discharge by the transverse section. In using floating balls, the air should be still, and the tract of water straight and uniform. The tachometer is to be held at several places of the space described by the floating ball, and it is also requisite for accuracy, that the diameter of the ball should be equal to that of the tachometer.

The second method of determination has several advantages when

the water in which the instrument is immersed is received into a gauge vessel. For this purpose, and especially for adjusting the hydrometer, it is well if the engineer can erect a proper hydraulic observatory, consisting of a vessel of efflux, a gauge reservoir, and a channel of communication between the two. With such an arrangement, we may impart to the water any arbitrary velocity, because we can not only regulate the entrance into the channel, but also the motion by means of boards placed in at pleasure. During observations we must keep the tachometer at different parts of the transverse section of the channel, measure the depth of this section by a scale, and, lastly, gauge the water running through in a definite time, in the lower reservoir (§ 372). We obtain the area F of the transverse profile by multiplication of the mean depth with the mean breadth, and the quantity of water Q is found from the mean transverse section G of the gauge measure, and the height (s) of the quantity which has flowed in during the time by the formula $Q = \frac{Gs}{t}$; but the mean velocity of the water: $v = \frac{Q}{F} = \frac{Gs}{Fl}$ follows from Q and F.

The corresponding number of revolutions u of the wheel is the mean of all the revolutions which are obtained when the instrument is immersed at different breadths and depths of the measured profile.

If from a series of experiments we have found the mean velocities v_1, v_2, v_3, &c., and the corresponding number of revolutions, we then obtain by substitution in the formula $v = v_0 + \alpha\, u$, or in the more correct one: $v = \alpha u + \sqrt{v_0^2 + \beta u^2}$ as many equations of condition for the constants v_0, α, β, as there have been observations made, and we may from these find the constants, if these equations are divided into as many groups as there are unknown constants, and these added together for as many equations of condition as are requisite for determining v_0, α, and also β when required.

Remark. If we adopt the more simple formula with 2 constants, we may then, after the method of least squares, put:

$$v_0 = \frac{\Sigma\,(y)^2\,\Sigma\,(x) - \Sigma\,(xy)\,\Sigma\,(y)}{\Sigma\,(x^2)\,\Sigma\,(y^2) - [\Sigma\,(xy)]^2} \text{ and } \alpha = \frac{\Sigma\,(x^2)\,\Sigma\,(y) - \Sigma\,(xy)\,\Sigma\,(x)}{\Sigma\,(x^2)\,\Sigma\,(y^2) - [\Sigma\,(xy)]^2},$$

where $x = \frac{1}{v}$ and $y = \frac{u}{v}$, and the sign Σ represents the sum of all successive similar values, for example, $\Sigma\,(x) = \frac{1}{v_1} + \frac{1}{v_2} + \frac{1}{v_3} + \dots$,

$$\Sigma\,(xy) = \frac{1}{v_1} \cdot \frac{u_1}{v_1} + \frac{1}{v_2} \cdot \frac{u_2}{v_2} + \frac{1}{v_3} \cdot \frac{u_3}{v_3} + \dots$$

Example. For a small tachometer the velocities are: 0,163; 0,205; 0,298; 0,366; 0,610 metres, the number of revolutions per second: 0,600; 0,835; 1,467; 1,805; 3,142 required to determine the constants corresponding to this wheel. From the formula given in the remark it follows, that:

$$\Sigma\,(x) = \frac{1}{0{,}163} + \frac{1}{0{,}205} + \dots = 18{,}740,\ \Sigma\,(y) = \frac{0{,}600}{0{,}163} + \dots = 22{,}759$$

$$\Sigma\,(x^2) = \left(\frac{1}{0{,}163}\right)^2 + \left(\frac{1}{0{,}205}\right)^2 + \dots = 82{,}846,\ \Sigma\,(y^2) = 105{,}223, \text{ and}$$

$$\Sigma\,(xy) = \frac{0{,}600}{(0{,}163)^2} + \frac{0{,}835}{(0{,}205)^2} + \dots = 80{,}961,$$

$$v_0 = \frac{105{,}223 \,.\, 18{,}740 - 80{,}961 \,.\, 22{,}759}{82{,}846 \,.\, 105{,}223 - (80{,}961)^2} = \frac{129{,}5}{2162} = 0{,}060 \text{ and}$$

$\alpha = \frac{368.3}{2162} = 0{,}1703$, hence for this instrument the formula $v = 0{,}060 + 0{,}1703\, u$.

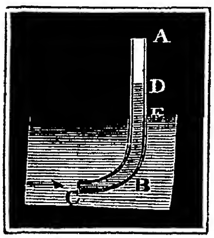
If in this we put $u = 0{,}6$, we then obtain·

$v = 0{,}060 + 0{,}102 = 0{,}162$; further, $u = 0{,}835$,
$v = 0{,}060 + 0{,}142 = 0{,}202$; further, $u = 1{,}467$,
$v = 0{,}060 + 0{,}249 = 0{,}309$, $u = 1{,}805$,
$v = 0{,}060 + 0{,}307 = 0{,}367$; lastly $u = 3{,}142$,
$v = 0{,}060 + 0{,}535 = 0{,}595$;

therefore, the calculated values agree very well with the observed.

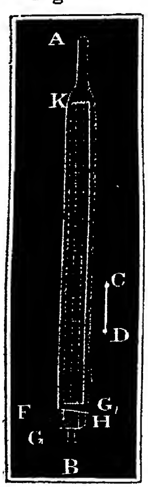
§ 380. *Pitot's Tube.*—Other hydrometers are not so satisfactory as the tachometer, for they either admit of less accuracy, or they are more complicated in their use. The most simple instrument of this kind is *Pitot's tube*. In its simplest form it consists of a bent glass tube ABC, Fig. 518, which is held in the water in such a manner that its lower part stands horizontally, and is opposed to the water. By the percussion of the water, a column of water is sustained in this tube, which stands above the level of the exterior fluid surface, and the elevation DE of this column is greater, the greater the percussion or the velocity of the water generating it; this elevation or difference of level may hence serve inversely for a measure of the velocity of the water. Let this elevation DE above the external surface of water $= h$, and the velocity $= v$, then $h = \frac{v^2}{2g\mu^2}$, where μ is a number derived from experiment, and we have inversely, $v = \mu \sqrt{2gh}$,

Fig. 518.



or more simply: $v = \psi \sqrt{h}$. To find the constant ψ, the instrument is immersed at a place in the water where the velocity v_1 is known; if the elevation is here $= h_1$, we then have the constant $\psi = \frac{v_1}{\sqrt{h_1}}$, which is to be applied in other cases, where the velocity is to be determined with this instrument.

Fig. 519.

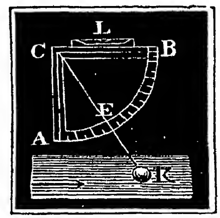


To facilitate the reading off of the height h, the instrument consists of two tubes, as shown in Fig. 519, and from the one a small tube F is directed against the stream, from the other two small tubes G and G_1 at right angles to the direction of the stream, both tubes are connected with a single cock H, by which the water can be retained in them. When the instrument is drawn out of the water, we may conveniently read off on a scale attached to both the tubes, the difference $CD = h$ of the two columns of water. That the water in the tube may not oscillate much, it is necessary to make the exterior orifices of the tubes narrow, and that the closing of them may take place quickly and safely; the cock is

provided with an arm and an even rod *HK*, which terminates above, near the handle of the instrument.

§ 381. *Hydrometric Pendulum.*—The hydrometric pendulum has been used in preference by Ximenes, Michelotti, Gerstner and Eytelwein for the measurement of the velocity of running water. This instrument consists of a quadrant *AB*, Fig. 520, divided into degrees and smaller parts, and a metallic or ivory ball *K* of from two to three inches diameter, suspended by a thread from the centre *C*, the velocity of the water is given by the angle *ACE*, at which the thread when stretched by the ball deviates from the vertical, when the plane of the instrument is brought into the direction of the stream, and the ball submerged in the water. As the angle rarely amounts to forty or more degrees, this instrument has often the form of a right angled triangle given to it, and the divisions made on its horizontal arm. For the placing of the index or zero line in the vertical, it is best to use a spirit level on the horizontal arm of the instrument, or the ball itself may serve for this purpose, by letting it be suspended out of the water, and the instrument revolve until the thread coincides with the zero line of the division.

Fig. 520.



For velocities under four feet we may use the ivory ball, but for greater velocities the hollow metal ball. On account of the constant undulations of the ball in the direction of the motion of the water, as also at right angles to the direction of the current, the reading off is somewhat difficult, and leaves a good deal of uncertainty, for which reason this instrument cannot be relied upon for the more exact numbers.

The dependence between the angle of deviation and the velocity of the water may be determined in the following manner when the ball is not very deeply immersed. From the weight *G* of the ball and from the impulse of the water $P = \mu\, Fv^2$, increasing simultaneously with the square of the velocity v and the section of the ball F, the resultant R, whose direction the thread assumes, follows, and is determined by the angle of deviation β, for which the $tang.\ \beta = \frac{P}{G} = \frac{\mu\, Fv^2}{G}$, hence also inversely:

$$v^2 = \frac{G\, tang.\ \beta}{\mu\, F}, \text{ and } v = \sqrt{\frac{G}{\mu F}} \cdot \sqrt{tang.\ \beta}, \text{ i. e. } v = \psi \sqrt{tang.\ \beta},$$

if ψ represents a co-efficient derived from experiment, which must be obtained before use, according to the above-mentioned instructions.

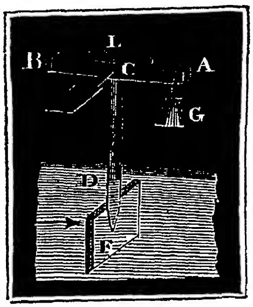
§ 382. *Rheometer.*—The remaining hydrometers, such as Lorgna's water lever, Ximenes's water vane, Michelotti's hydraulic balance, Brünning's tachometer, Poletti's rheometer, are more complicated in their use, and not altogether to be relied on. The principle of all these instruments is the same, they are composed of a surface of impulse and a balance, and the last serves for the purpose of giving the

percussion P of the water against the former, but since this $= \mu \, Fv^2$, we then have inversely:

$$= \sqrt{\frac{P}{\mu F}} = \psi \sqrt{P},$$ where ψ denotes a constant deduced from experiment dependent on the magnitude of the surface of impact F.

The *rheometer*, which was lately proposed by Poletti, and does not materially differ from the hydrometric balance of Michelotti, consists of a lever AB, Fig. 521, turning about a fixed axis C, and an arm CD to which the surface of impulse, or according to Poletti, a mere impulse-staff is screwed. To maintain equilibrium with the percussion of the water against the surface, the boxes suspended at the extremity A of the lever are loaded with weight or shot, and to put the empty balance in equilibrium in still water, a counterpoise is placed at B, which makes up the outermost end of the arm CB. From the weight put on G, the impulse P is found by means of the arm $CA = a$ and $CF = b$ from the formula $Pb = Ga$, whence, therefore,

Fig. 521.



$$P = \frac{a}{b} \, G, \text{ and } v = \sqrt{\frac{P}{\mu F}} = \sqrt{\frac{a \, G}{\mu \, b \, F}} = \psi \sqrt{G},$$

where ψ is a constant derived from experiment.

Remark. With respect to the last hydrometer, ample details will be found in Eytelwein's "Handbuch der Mechanik fester Korper und der Hydraulik;" further, in Gerstner's "Handbuch der Mechanik," vol. 2; in Brünning's "Treatise on the velocity of running water;" in Venturoli's "Elementi di Meccanica e d'Idraulica," vol. 2. Concerning Poletti's hydrometer, we must refer to Dingler's "Polytechn. Journal," vol. 20, 1826. The hydrometer described in Stevenson's treatise on Marine Surveying and Hydrometry is the tachometer of Woltmann, *see* Dingler's "Journal," vol. 65, 1842.

CHAPTER IX.

ON THE IMPULSE AND RESISTANCE OF FLUIDS.

§ 383. *Impulse and Resistance of Water.*—Water or any other fluid imparts a shock to a rigid body, when it meets it in such a manner that its condition of motion is thereby altered. The resistance which water opposes to the motion of a body, does not essentially differ from impulse. The investigation of these two forms the third principal division of hydraulics. We distinguish from each other:

1. The impulse of an isolated stream.
2. The impulse of a limited stream.
3. The impulse of an unlimited stream.

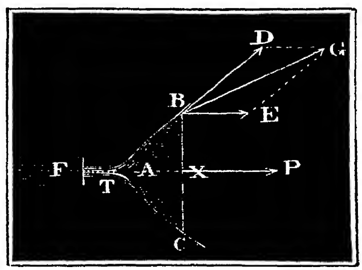
An impulse of the first kind takes place when a body, for instance, the float board of an over-shot water-wheel, is opposed to a stream of water issuing from a reservoir; an impulse of the second kind occurs where water, in a canal or in a water-course, impinges against a body which entirely fills up its transverse section, as for instance, against the float board of an under-shot wheel; the third kind, lastly, presents itself, when running water strikes against a body immersed in it, whose transverse section is only a very small part of that of the current of water, as, for instance, against the float boards of a floating mill-wheel.

We must distinguish the impulse of water against a body at rest and against a body in motion, and further, the impulse against a curved and against a plane surface, and in this last again, between the perpendicular and the oblique impulse.

Let us consider at once the general case, namely, the impulse of an isolated stream against a surface of rotation which moves in its proper axis, and in the direction of motion of the stream.

§ 384. *Impact of Isolated Streams.*—Let BAC, Fig. 522, be a surface of rotation, AX its axis, and FA a fluid stream meeting it in this direction. Let the velocity of the water $= c$, that of the surface $= v$, and the angle BTX, which the tangent DT at the extremity B of the generating curve or of each of the filaments of water BT leaving the surface, includes with the direction of the axis $BE = \alpha$; lastly, let us further assume that the water in running off from the surface loses nothing in *vis viva* by friction. The water strikes against the surface with the relative velocity $c - v$, and hence leaves the surface with this, and therefore quits it in the tangential directions TB, TC, &c. From the tangential velocity $BD = c - v$, and the velocity of the axis $BE = v$, the absolute velocity $BG = c_1$ of the water after impinging against the surface is found by the known formula:

Fig. 522.



$$c_1 = \sqrt{(c-v)^2 + 2(c-v)\, v \cos. \alpha + v^2}.$$

But now a quantity of water Q is able to produce by virtue of its *vis viva* the mechanical effect $\frac{c^2}{2g} \cdot Q\gamma$, if its velocity c is fully imparted; accordingly the residuary effect of the water:

$= \frac{c_1^2}{2g} \cdot Q\gamma$; consequently the mechanical effect distributed over the surface is:

$$Pv = \frac{c^2}{2g} Q\gamma - \frac{c_1^2}{2g} Q\gamma = \frac{c^2 - c_1^2}{2g} \cdot Q\gamma.$$

$$= \frac{[c^2 - (c-v)^2 - 2(c-v)\, v \,.\, \cos. \alpha - v^2]}{2g} Q\gamma$$

$$= \frac{2\,c\,v - 2\,v^2 - 2\,(c - v)\,v\,cos.\,\alpha}{2g}\,Q\gamma,\ i.e.$$

$$Pv = (1 - cos.\,\alpha)\,\frac{(c - v)\,v}{g}\,Q\gamma,$$

and the force or the impulse of the water in the direction of its axis is:

$$P = (1 - cos.\,\alpha)\,\frac{(c - v)}{g}\,Q\gamma.$$

If the surface meets the water with the velocity v, we then have:

$$P = (1 - cos.\,\alpha)\,.\,\frac{(c + v)}{g}\,Q\gamma,$$

and if this is without motion, therefore, $v = 0$, the impulse or hydraulic pressure of the axis comes out:

$$P = (1 - cos.\,\alpha)\,\frac{c}{g}\,.\,Q\gamma.$$

It follows from this, that the impulse of one and the same mass of water under otherwise similar circumstances is proportional to the relative velocity $c \mp v$ of the water.

From the area F of the transverse section of the fluid stream, it follows that the quantity discharged is $Q = Fc$; hence

$$P = (1 - cos.\,\alpha)\,\frac{(c \mp v)\,c}{g}\,F\gamma;$$

and for $v = 0$:

$$P = (1 - cos.\,\alpha)\,\frac{c^2}{g}\,F\gamma.$$

For an equal transverse section of the stream, the impulse against a surface at rest increases therefore as the square of the velocity of the water.

§ 385. *Impulse against Plane Surfaces.*—The impulse of one and the same fluid stream depends principally on the angle α, under which the water, after the impulse, leaves the axis; it is nothing if this angle $= 0$; and, on the other hand, a maximum, namely, $= 2\,\frac{(c \mp v)}{g}\,Q\,\gamma$, if this angle is 180°, therefore its cosine $= -1$, where the water, as represented in Fig. 523, leaves the surface in a

Fig. 523.

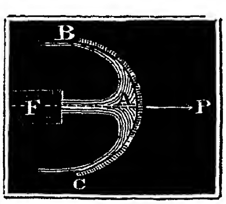
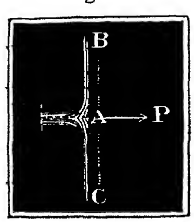


Fig. 524.



direction opposite to that in which it impinges. This is generally greater for concave surfaces than for convex, because the angle is there

oblique, therefore the cosine negative and $1 - cos. \alpha$ becomes $1 + cos. \alpha$.

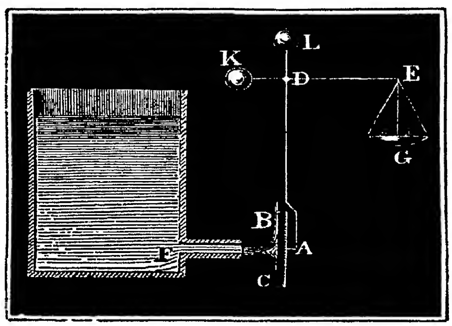
Most frequently the surface, as represented in Fig. 524, is plane, and hence $\alpha = 90°$, therefore $cos. \alpha = 0$, and the impulse $P = \frac{(c \mp v)}{g} \cdot Q\gamma$; for a surface at rest:

$$P = \frac{c}{g} Q\gamma = \frac{c^2}{2g} F\gamma = 2 \cdot \frac{c^2}{2g} F\gamma = 2\,F\,h\,\gamma.$$

The normal impulse of water against a plane surface is therefore equivalent to the weight of a column of water which has for base the transverse section F of the stream, and for altitude, twice the height due to the velocity $2\,h = 2 \cdot \frac{c^2}{2g}$.

The experiments made on this subject by Michelotti, Vince, Langsdorf, Bossut, Morosi, and Bidone, have nearly led to the same results when the transverse section of the impinged surface was at least six times as great as that of the stream, and when this surface was twice as far from the plane of the orifice as the thickness of the stream. The apparatus which was used for this purpose consisted of a lever, similar to that of Poletti's rheometer, which received upon one side the impulse of the water, and whilst its other side was kept in equilibrium by weights. The instrument which Bidone made use of is represented in Fig. 525. *BC* is the surface impinged on by the stream *FA*, *G* is the scale-pan for the reception of the weights, *D* the axis of rotation, *KL* counter-weights.*

Fig. 525.



§ 386. *Maximum Effect of Impulse.*—The mechanical effect of impulse:

$$Pv = (1 - cos. \alpha) \frac{(c - v)\,v}{g} Q\gamma$$

depends principally on the velocity v of the impinged surface; it is,

* The latest and most extensive experiments on the percussion of water are those of Bidone. *See* "Memorie de la Reale Accademia delle Scienze di Torino," vol. 40, 1838. They were performed with a velocity of at least 27 feet, and on brass plates of from 2 to 9 inches diameter. In general, Bidone found that the normal impulse against a plane surface was somewhat greater than $2\,F\,h\,\gamma$, yet this variation is perhaps to be attributed to an augmentation of the leverage which is produced by the falling back of the water. *See* Duchemin's "Recherches expérim. sur les lois de la résistance des fluides." When the impinged surface was quite near the orifice, Bidone found that P was only $1{,}5\,F\,h\,\gamma$; when, further, the surface had a transverse section equal to that of the stream, in which case the water only deviated by an acute angle α, then, after Du Buat and Langsdorf, P was only $= F\,h\,\gamma$. Lastly, it has been deduced by Bidone and others that the impulse is in the first moment nearly as great again as the permanent impulse.

for example, nothing, not only for $v = c$, but also for $v = 0$; hence there is a velocity for which the effect of the impulse is a maximum. It is manifest that it only depends on $(c—v)\,v$ becoming a maximum. If we consider c as half the perimeter of a rectangle, and v as its base, we have then its height $= c—v$ and its area $= (c—v)\,v$. But of all rectangles the square is that which has for a given perimeter $2\,c$ the greatest area, hence also $(c—v)\,v$ is a maximum, when $c—v = v$, i. e. $v = \frac{c}{2}$, *and we therefore obtain the maximum value of the mechanical effect of the impulse when the surface moves from it with half the velocity of the water, and indeed*

$$Pv = (1—cos.\,\alpha)\,.\,\tfrac{1}{2}\,.\,\frac{c^2}{2g}\,.\,Q\gamma = (1—cos.\,\alpha)\,.\,\tfrac{1}{2}\,Qh\gamma.$$

If now $\alpha = 180^{\circ}$, and if, therefore, the motion of the water be reversed by the impulse, we then have the effect equal to $2\,.\,\frac{1}{2}\,Qh\gamma = Qh\gamma$. But if $\alpha = 90^{\circ}$, *i. e.* if it impinges against a plane surface, this effect is then only $\frac{1}{2}\,Qh\gamma$, therefore, in the last case, the half only of the whole disposable effect, or that which corresponds to the *vis viva* of the water, is gained or brought to bear upon the surface.

Examples.—1. If a stream of water, of 40 square inches transverse section, delivers a quantity of 5 cubic feet per second, and strikes normally against a plane surface, and escapes with a 12 feet velocity, the effect of impulse is then:

$$P = \frac{(c-v)}{g}\,Q\gamma = \left(\frac{5\,.\,144}{40} - 12\right).\,0{,}031\,.\,5\,.\,62.5 = 6\,.\,0{,}031\,.\,312{,}5 = 58{,}12 \text{ lbs.},$$

and the mechanical effect brought to bear upon the surface $Pv = 58{,}12 \times 12 = 697{,}44$ ft. lbs. The greatest effect is for $v = \frac{c}{2} = \frac{1}{2}\,.\,\frac{5\,.\,144}{40} = 9$ feet, and indeed:

$$= \tfrac{1}{2}\,.\,\frac{c^2}{2g}\,.\,Q\,\gamma = \tfrac{1}{2}\,.\,18^2\,.0{,}0155\,.\,5\,.\,62{,}5 = 81\,.\,0{,}0155\,.\,62{,}5 = 784{,}68 \text{ ft. lbs.};$$

the corresponding impulse, or hydraulic pressure $= \frac{784{,}68}{9} = 87{,}18$ lbs.—2. If a stream FA, Fig. 526, of 64 square inches section, strikes with a 40 feet velocity against an immovable cone, having an angle of convergence $BAC = 100^{\circ}$, then is the hydraulic pressure in the direction of the stream:

$$P = (1 - cos.\,\alpha)\,\frac{c}{g}\,Q\,\gamma = (1 - cos.\,50^{\circ})\,40\,.\,0{,}031\,.\,\frac{64}{144}\,.\,40\,.\,62{,}5$$

$$= (1—0{,}64279)\,.\,1\,.\,24\,.\,\frac{10000}{9} = 0{,}35721\,.\,1377{,}7 = 492\,.\,13 \text{ lbs.}$$

Fig. 526.

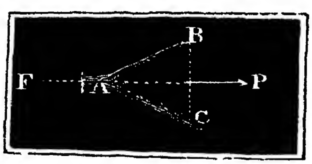
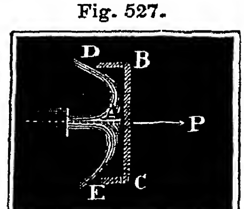


Fig. 527.

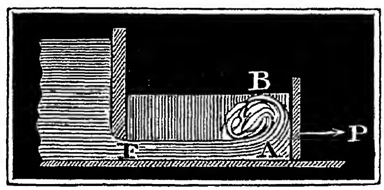


§ 387. *Impulse of a Limited Stream.*—If we add borders BD, CE, to the perimeter of a plane surface BE, Fig. 527, which project from the side impinged upon by the water, then will the water deviate from its direction at an obtuse angle, in a similar manner as from

concave surfaces, and hence the impulse will be greater than for plane surfaces. The effect of this impulse depends principally on the height of the border and the ratio of the transverse section between the stream and the part confined. In an experiment, where the stream was 1 inch thick, the cylindrical enclosure 3 inches wide and $3\frac{1}{2}$ lines deep, the water ran off almost in a reversed direction, and the impulse amounted to $3{,}93 \frac{c^2}{2g} F\gamma$; in every other case this force was less. In consequence of the friction of the water at the surface and the sides, the theoretical maximum value never reaches $4 \frac{c^2}{2g} F\gamma$.

In the impulse of a limited stream FAB, Fig. 528, a rising at the edges takes place; this rising occupies only a portion of the perimeter, and extends itself, on the other hand, simultaneously to the impinged surface and the fluid stream. The impinging water takes the direction of the unbordered portion of the perimeter, and here, therefore, becomes deflected 90 degrees, whence the formula above found for the isolated stream $P = \frac{(c - v)}{g} Q\gamma$ holds good; yet this may also be deduced in the following manner. If we assume that the velocity c of the arriving water by the impulse against its surface is changed into the velocity v of the surface, we may then also assume that a loss of mechanical effect $\frac{(c - v)^2}{2g} Q\gamma$ (similar to that in § 337), expended in the division of the water, is connected with it. But now the effect due to the *vis viva* of the arriving water $= \frac{c^2}{2g} Q\gamma$ and to that of the water going on $= \frac{v^2}{2g} Q\gamma$, hence it follows that the mechanical effect imparted to the surface is:

Fig. 528.



$$Pv = [c^2 - (c - v)^2 - v^2] \frac{1}{2g} Q\gamma = \frac{(c - v)\,v}{g} Q\gamma.*$$

§ 388. *Oblique Impulse.*—In oblique impulse against a plane surface, we must distinguish whether the water flows away in one, two, or in all directions in the plane. If, as in the impact of limited water, the surface AB, Fig. 529, is confined at three sides, so that the water can run off only in one direction, we have then the hydraulic pressure against the surface in the direction of the stream $P = (1 \cos. \alpha) \frac{(c - v)}{g} Q\gamma$.

* This formula will be found applicable hereafter, when we come to the theory of water-wheels.

Fig. 529.

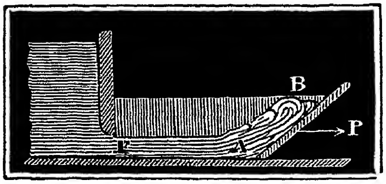
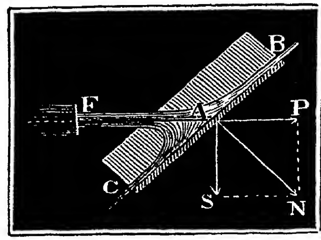


Fig. 530.



But if the impinged plane BC, Fig. 530, is only bordered on two oppositely situated sides, the stream then divides itself into two unequal portions; the greater portion Q_1 takes the small deflexion α, and the lesser Q_2, the greater deflexion $180 - \alpha$; hence, the whole impulse in the direction of the stream is:

$$P = (1 - \cos.\,\alpha)\,.\,\frac{c - v}{g}\,Q_1\gamma + (1 + \cos.\,\alpha)\,.\,\frac{c - v}{g}\,Q_2\,\gamma = \left(\frac{c - v}{g}\right)\gamma\,[(1 - \cos.\,\alpha)\,Q_1 + (1 + \cos.\,\alpha)\,Q_2].$$

Now the equilibrium of the two portions of the stream requires that the pressures

$$\frac{(c - v)}{g}\,\gamma\,(1 - \cos.\,\alpha)\,Q_1 \text{ and } \frac{(c - v)}{g}\,\gamma\,(1 + \cos.\,\alpha)\,Q_2$$

between them should be equal; hence, also:

$$(1 - \cos.\,\alpha)\,Q_1 = (1 + \cos.\,\alpha)\,Q_2, \text{ or since } Q_1 + Q_2 = Q,$$

$$(1 - \cos.\,\alpha)\,Q_1 = (1 + \cos.\,\alpha)\,(Q - Q_1), \ i.\,e.,$$

$$Q_1 = \left(\frac{(1 + \cos.\,\alpha)}{2}\right) Q, \text{ and } Q_2 = \left(\frac{1 - \cos.\,\alpha}{2}\right) Q,$$

so that the whole impulse in the direction of the stream is finally:

$$P = \frac{(c - v)}{g}\,\gamma\,.\,2\,(1 - \cos.\,\alpha)\,\frac{(1 + \cos.\,\alpha)\,Q}{2} = \frac{(c - v)\gamma}{g}\,(1 - \cos.\,\alpha^2)\,Q,$$

$$i.\,e.,\ P = \frac{c - v}{g}\,\sin.\,\alpha^2\,.\,Q\,\gamma.$$

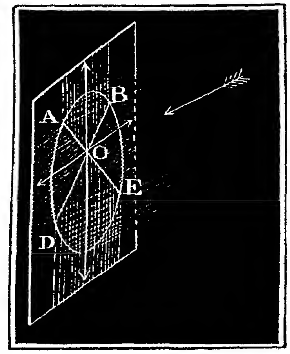
Besides the *parallel impulse* P, acting in the direction of the stream, we distinguish, further, the *lateral impulse* S, acting at right angles to the direction of the stream, and the *normal impulse* N, composed of these two, and at right angles to the surface. In every case $P = N \sin.\,\alpha$, and $S = N \cos.\,\alpha$; hence, inversely,

$$N = \frac{P}{\sin.\,\alpha} = \frac{c - v}{2g}\,\sin.\,\alpha\ Q\gamma \text{ and } S = \frac{c - v}{2g}\,\sin.\,2\,\alpha\,.\,Q\gamma.$$

The normal impulse, therefore, increases as the sine, the parallel impulse as the square of the sine of the angle of incidence, and the lateral impulse as double the same angle. Lastly, if the inclined surface impinged on is not bordered, then the water can spread over it in all directions; the impulse is then greater, because of all the angles by which the filaments of water are deflected, α is the least; and hence, each filament which does not move in the normal plane, exerts a

greater pressure than the filament in this plane. Let us assume tha a portion Q_1 corresponding to the sectors AOB and DOE, Fig. 531, is deflected by the angles α and $180^\circ - \alpha$, and another Q_2, corresponding to the sectors AOD and BOE, by 90°, and that both portions exert a parallel impulse, we may then put:

Fig. 531.



$$P = \frac{c-v}{g}\, Q_1 \gamma \sin.\alpha^2 + \frac{c-v}{g}\, Q_2 \gamma,\ Q_1 \sin.\alpha^2$$

$= Q_2$, and $Q_1 + Q_2 = Q$; hence it follows, that $Q_1 (1 + \sin.\alpha^2) = Q$, and the whole parallel impulse $P =$

$$\left(\frac{c-v}{g}\right) \frac{2\, Q\gamma \sin.\alpha^2}{1 + \sin.\alpha^2} = \frac{2 \sin.\alpha^2}{1 + \sin.\alpha^2} \cdot \frac{c-v}{g} \cdot Q\gamma.$$

Although this hypothesis is only approximately correct, it tolerably well agrees, nevertheless, with the latest experiments of Bidone.

§ 389. *Action of an Unlimited Stream.*—If a body moves progressively in an unlimited fluid, or if a body is put into a fluid which is in motion, it then suffers a pressure which is dependent on the form and dimensions of this body, as well as on the density and on the velocity of the one or the other mass, and in the one case is called the *resistance*, and in the other the *impulse* of the fluid. This hydraulic pressure arises principally from the inertia of the water, whose condition of motion is altered by striking against the solid body, and also, further, from the force of cohesion of the particles of water, which are hereby partially separated from one another, or pushed aside. If a body AC moves against running water, Fig. 532, it pushes away

Fig. 532.

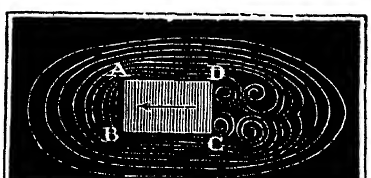
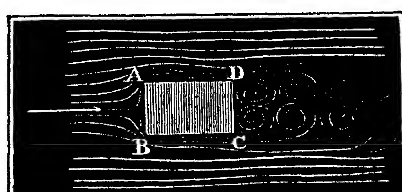


Fig. 533.



before it a certain quantity with an augmented pressure. Whilst this mass of water, by the further advance of the body, always increases on the one side, on the other a constant flowing away takes place, while the particles lying near the anterior surface assume a motion in the direction of this surface. If the moving mass of water strikes against a body at rest, Fig. 533, then is there likewise an increased pressure produced in front of it, which causes the particles before the body to deviate from their original direction, and to run off at the surface AB. When these particles have reached the limits of the surface, they then turn and flow away by the lateral surfaces until they come to the back, when they then again immediately unite. but

assume an eddying motion. It is manifest that the general circumstances of motion of the particles surrounding the body are the same in the impact of moving water as in the resistance of a body moving in water, except that in the eddies a difference so far takes place, that with short bodies the eddy in the latter case occupies a less space than in the former. In both cases the velocity of the particles increases more and more from the middle of the anterior surface to its limits, attains its maximum at the commencement of the lateral surfaces, where, for the most part, a contraction takes place, gradually diminishes in the water which passes away at the sides, and lastly, attains its minimum when the water reaches the back and passes into a whirling motion.

§ 390. *Theory of Impulse and Resistance.*—The normal pressure varies at different points of the body; it is greatest at the middle of the anterior, and least at the middle of the posterior surface, and, next to that, at the parts of the sides nearest this; because, in respect to the body, there is at the one place rather a flow to, and at the other a flow from these surfaces. If the body be symmetrical, as we shall suppose it to be, with respect to the direction of motion, then the aggregate pressures in this direction counteract each other, and hence only the pressures in the direction of motion are to be taken into account. But now the pressures on the posterior surface are opposed to those on the anterior; hence the *resultant impulse or resistance of the water may be equated to the difference of pressure of the anterior and posterior surfaces.*

If we cannot assign the amount of these pressures *à priori*, we may, nevertheless, from the great similarity of the circumstances to the impulse of isolated streams, assume that at least the general law for the impulse of unlimited water does not differ from that of the impulse of isolated streams. If, therefore, F is the area of a surface, which is impinged on by an unlimited current whose density is γ, with a velocity v, then the corresponding impulse or hydraulic pressure may be put $P = \zeta \dfrac{v^2}{2g} F\gamma$, where ζ represents a number deduced from experiment, dependent on the form of the surface. But this expression is not only applicable to action against the anterior, but also to that against the posterior surface, only that in this last, when the water has a tendency to flow away, it consists of a draught or negative pressure. If now $Fh\gamma$ is the hydrostatic pressure (§ 276) against the front and back surface of a body, the whole pressure against the front is: $P_1 = Fh\gamma + \zeta_1 . \dfrac{v^2}{2g} F\gamma$, and that against the back: $P_2 = Fh\gamma - \zeta_2 . \dfrac{v^2}{2g} F\gamma$, and the resultant impulse or resistance of the water is then found:

$$P = P_1 - P_2 = (\zeta_1 + \zeta_2) . \frac{v^2}{2g} F\gamma = \zeta . \frac{v^2}{2g} F\gamma, \text{ if } \zeta_1 + \zeta_2 = \zeta.$$

This general formula for the impulse of unlimited water is applicable to the percussion of the wind or to the resistance of the air. Besides the

difference of aërodynamic pressure at the front and back, there is further a difference of aërostatic pressure, because the air in front, in consequence of its greater elasticity, has a greater density (γ) than that at the back. For this reason, in high velocities, as those of cannon-balls, the co-efficient of the resistance of air is greater than that of water.

Remark.—The adhesion of a certain quantity of air or water to the body, is a peculiar phenomenon of the impulse or resistance of an unlimited medium (water or air), whose influence is particularly remarkable in the variable motion of bodies, as, for example, in the oscillations of the pendulum. For a ball, the air or water adhering to the moving body is equal to 0,6 of the volume of the ball. For a prismatic body moved in the direction of its axis, the ratio of this volume $= 0{,}13 + 0{,}705 \frac{\sqrt{F}}{l}$, where l is the length, and F the transverse section of the body. These relations, discovered by Du Buat, have been fully confirmed by the later observations of Bessel, Sabine, and Baily.

§ 391. *Impulse and Resistance against Surfaces.*—The co-efficient of resistance ζ, or the number with which the height due to the velocity is to be multiplied to obtain the height of a column of water measuring this hydraulic pressure, varies for bodies of different figures, and only for plates which are at right angles to the direction of motion is it nearly a definite quantity. According to the experiments of Du Buat and those of Thibault, we may put $\zeta = 1{,}85$ for the impulse of air or water against a plane surface at rest, and, on the other hand, assume, but with less accuracy, for the resistance of air or water against a surface in motion $\zeta = 1{,}40$. In both cases, about two-thirds of the whole effect are expended on the front, and one-third on the back. The resistance which the air opposes to a surface revolving in a circle, has been found by Borda, Hutton, and Thibault to vary a good deal, but may be expressed by a mean of $\zeta = 1{,}5$. If the surface does not stand at right angles to the direction of the motion, but makes with it an acute angle α, we may then, with Duchemin, substitute for ζ, $\frac{2\,\zeta\, sin.\, \alpha^2}{1 + sin.\, \alpha^2}$ with tolerable correctness.

The impulse and resistance of unlimited media are also augmented when the surfaces are hollowed out or have projecting edges at their perimeters, but we have arrived at no general results on this subject.

Example. If the wind impinges with a 20 feet velocity against a firmly fixed wind-mill wheel, which consists of four wings, of which each has an area of 200 square feet and 75° inclination to the direction of the wind, then is the impinging force of the wind in its direction, or in that of the axis of the wheel:

$$P = 1{,}85 \cdot \frac{2\,(sin.\,75)^2}{1 + (sin.\,75)^2} \cdot \frac{20^2}{2g} \cdot 4 \cdot 200 \cdot 0{,}081 = 1{,}85 \cdot 0{,}965 \cdot 6{,}21 \cdot 800 \cdot 0{,}081$$
$= 718{,}4$ ft. lbs., when the density of the wind is (from § 301) taken at 0,081 lbs.

Remark. Views, with respect to the impulse and resistance of unlimited fluids, entirely at variance with these, are put forward in the above-mentioned work of Duchemin. It is there maintained, for instance, that the impulse and resistance against the front surface of a thin plate amounts to $2 \cdot \frac{v^2}{2g} F h$, and is not negative at the back, that the impulse $= 0{,}136 \frac{v^2}{2g} F\gamma$, and the resistance $= 0{,}746 \frac{v^2}{2g} F\gamma$. It would be too circumstantial here to give a detail of the reasons why the author cannot agree with the views of Duchemin, but more with reference to this will be found in Poncelet's "Introduction à la mécanique industrielle," 2d edition, 1841.

§ 392. *Impulse and Resistance to Bodies.*—The impulse and resistance of water to prismatic bodies, whose axis coincides with the direction of motion, diminishes when the length of the body is considerable. From the experiments of Du Buat and Duchemin, the impulse of the front surface is invariable, and only the effect against the back surface variable. To this corresponds the co-efficient $\zeta_1 = 1{,}186$, for the total effect, however, with the relative lengths

$$\frac{l}{\sqrt{F}} = 0, \quad 1, \quad 2, \quad 3,$$

$$\zeta = 1{,}86;\ 1{,}47;\ 1{,}35;\ 1{,}33.$$

For still greater ratios between the length l and the mean breadth $\sqrt{F}$ of the body ζ diminishes, owing to the friction of the water at the lateral surfaces of the body. From the resistance of the water, reverse relations take place. Here, from Du Buat, for the effect on the front surface, $\zeta_1 = 1$ invariably; for the total effect, however, with

$$\frac{l}{\sqrt{F}} = 0, \quad 1, \quad 2, \quad 3,$$

$\zeta = 1{,}25;\ 1{,}28;\ 1{,}31;\ 1{,}33$, so that, for a prism which is 3 times as long as broad, the impulse is the same as the resistance.

The experiments undertaken by Borda, Hutton, Vince, Desaguiliers, Newton, and others, with angular and with round bodies, leave still much uncertainty. In what relates to spheres, it appears that for moderate velocities the mean co-efficient for motion in air or water $= 0{,}6$. For a greater velocity and for motion in air, according to Robins and Hutton, for the velocities

$$v = 1, \quad 5, \quad 25, \quad 100, \quad 200, \quad 300, \quad 400, \quad 500, \quad 600 \text{ metr.}$$

$$\zeta = 0{,}59;\ 0{,}63;\ 0{,}67;\ 0{,}71;\ 0{,}77;\ 0{,}88;\ 0{,}99;\ 1{,}04;\ 1{,}10.$$

Duchemin and Piobert have given particular formulæ for the rate of increase of these co-efficients.

For the impulse of water against a sphere, Eytelwein found $\zeta = 0{,}7886$.*

Example. If, according to Borda, we put the resistance and impact at right angles to the axis of a cylinder at half as great as that against a parallelopiped which has the same dimensions, we then obtain for the resistance $\zeta = \frac{1}{2} \cdot 1{,}28 = 0{,}64$ and the impact $= \frac{1}{2} \cdot 1{,}47 = 0{,}735$. If we apply these values to the human body, whose section has an area of some 7 square feet, we then find for the resistance and impulse of air against it, the values:

$P = 0{,}64 \cdot 0{,}0155 \cdot 7 \cdot 0{,}081\, v^2 = 0{,}00562\, v^2$, and

$P = 0{,}735 \cdot 0{,}0155 \cdot 7 \cdot 0{,}081\, v^2 = 0{,}00646\, v^2$. Hence the resistance of air for a velocity of 5 feet is only $0{,}00562 \cdot 25 = 0{,}1405$ lbs.; and the corresponding mechanical effect per second $= 5 \cdot 0{,}1405 = 0{,}70$ ft. lbs.; for a velocity of 10 feet this resistance is four times, and the effect expended eight times as great, and for a velocity of 15 feet, the resistance is 9 times and the effect 27 times as great as for a 5 feet velocity. If a man, with a 5 feet velocity, moves against wind having a 50 feet velocity, he has then a resistance $0{,}00646 \cdot 55^2 = 19{,}54$ lbs. to overcome, corresponding to the relative velocity $50 + 5 = 55$ feet, and thereby to produce the mechanical effect of $19{,}54 \cdot 5 = 97{,}7$ ft. lbs. (English.)

* Poncelet, in his work above cited, and Duchemin and Thibault in their "Recherches expérimentales," have treated very fully of these circumstances. In the Second Part we shall treat of the *resistance to floating bodies*, especially to ships, &c., as also the impact of the wind on wheels, &c.

§ 393. *Motion in Resisting Media.*—The laws of the motion of a body in a resisting medium are rather complex, because we have here to deal with a variable force, *i. e.*, one increasing with the square of the velocity. From the force P_1 which urges the body forward, and from the resistance $P_2 = \zeta \cdot \frac{v^2}{2g} F\gamma$, which the medium opposes to the motion, the motive force is:

$$P = P_1 - P_2 = P_1 - \zeta \cdot \frac{v^2}{2g} F\gamma,$$

but since the mass of the body $= M = \frac{G}{g}$, the accelerating force is:

$$p = \frac{P}{M} = \left(P_1 - \zeta \frac{v^2}{2g} F\gamma\right) \div M = \left(\frac{P_1 - \zeta \frac{v^2}{2g} F\gamma}{G}\right) \cdot g,$$

or if we represent $\frac{F\gamma}{2gP_1}$ by $\frac{1}{w^2}$.

$p = \left[1 - \zeta \left(\frac{v}{w}\right)^2\right] \frac{P_1}{G} g$. But the velocity v is accelerated in the instant of time τ by $\varkappa = p\,\tau$, hence:

$\varkappa = \left[1 - \zeta \left(\frac{v}{w}\right)^2\right] \frac{P_1}{G} g\,\tau$, and inversely:

$$\tau = \frac{G_1}{P_1} \cdot \frac{\varkappa}{g\left[1 - \zeta \left(\frac{v}{w}\right)^2\right]}.$$

Now to find the time corresponding to a given change of velocity, let us divide the difference $v_n - v_0$, of the final and initial velocity into n parts, let any such part $\frac{v_n - v_0}{n} = \varkappa$, and let us calculate the velocities:

$$v_1 = v_0 + \varkappa,\ v_2 = v_0 + 2\,\varkappa,\ v_3 = v_0 + 3\,\varkappa, \text{ \&c.},$$

and substitute these values in the formula of Simpson. In this manner, by taking four parts we shall obtain the time sought

$$1.\ t = \frac{G}{P_1} \cdot \frac{v_n - v_0}{12g} \left(\frac{1}{1 - \zeta \left(\frac{v_0}{w}\right)^2} + \frac{4}{1 - \zeta \left(\frac{v_1}{w}\right)^2} + \right.$$

$$\left. + \frac{2}{1 - \zeta \left(\frac{v_2}{w}\right)^2} + \frac{4}{1 - \zeta \left(\frac{v_3}{w}\right)^2} + \frac{1}{1 - \zeta \left(\frac{v_4}{w}\right)^2}\right).$$

Further, the small space described in any instant τ (§ 19), is $\sigma = v\,\tau$, or since $\tau = \frac{\varkappa}{p}$, $\sigma = \frac{v\varkappa}{p}$, therefore,

$\sigma = \frac{v\varkappa}{1 - \zeta \left(\frac{v}{w}\right)^2} \cdot \frac{G}{P_1 g}$. By the application of Simpson's rule, we

shall now find the space which is described while the velocity v passes into that of v_n.

$$2.\ s = \frac{G}{P_1} \cdot \frac{v_n - v_0}{12\,g} \left(\frac{v_0}{1 - \zeta \left(\frac{v_0}{w}\right)^2} + \frac{4v_1}{1 - \zeta \left(\frac{v_1}{w}\right)^2} + \right.$$

$$\left. + \frac{2v_2}{1 - \zeta \left(\frac{v_2}{w}\right)^2} + \frac{4v_3}{1 - \zeta \left(\frac{v_3}{w}\right)^2} + \frac{v_4}{1 - \zeta \left(\frac{v_4}{w}\right)^2} \right).$$

Of course the accuracy is greater, when we take six, eight, or more parts. This formula takes into account the variability of the co-efficients of resistance, which in considerable velocities is necessary. For the free descent of bodies in air or water $P_1 = G$, and for motion on a horizontal plane $P_1 = 0$, is more correctly equal to the friction $f\,G$. Since this is a resistance, we have then to introduce it as negative into the calculation, whence

$$P = -(P_1 + P_2), \text{ and } p = -\left[1 + \zeta \left(\frac{v}{w}\right)^2\right] \frac{P_1}{G} g.$$

As it cannot be a question here of an increase, but only of a diminution of velocity, we have then to substitute in the above formula $v_0 - v_n$ for $v_n - v_0$.

In the case, where the body is urged by a force, by its weight for instance, the motion approximates more and more to a uniform one, so that after the lapse of a certain time, it may be considered as such, although not so in reality. The accelerating force $p = 0$, when

$$\zeta \cdot \frac{v^2}{2g} F\gamma = P_1, \text{ when, therefore, } v = \sqrt{\frac{2g\,P_1}{\zeta\,F\gamma}}.$$

The velocity of a falling body approximates, therefore, to this limit more and more, without ever actually attaining it.

Example. Piobert, Morin, and Didion found, for a parachute whose depth was 0,31 that of the diameter of its opening $\zeta = 1{,}94 \cdot 1{,}37 = 2{,}66$. Hence, from what height in Prussian feet will a man, of 150 lbs. weight, be able to descend with a similar parachute, of 10 lbs. weight and 60 square feet transverse section, without acquiring a greater velocity than that which he would have acquired by jumping from a 10 feet height, without a parachute? The last velocity is $v = 7{,}906\sqrt{10} = 25$ feet, the force is $P_1 = G = 150 + 10 = 160$ lbs., the surface $F = 60$ square feet, the density $\gamma = 0{,}0859$, and the co-efficient of resistance $\zeta = 2{,}66$, hence:

$\frac{1}{w^2} = \frac{60 \cdot 0{,}0859}{62{,}5 \cdot 160} = 0{,}000515$, and $\zeta \cdot \frac{v^2}{w^2} = 2{,}66 \cdot 0{,}000515 \cdot 25^2 = 0{,}85625$. If, therefore, we take 6 parts, we then obtain for these:

$1 - \zeta \cdot \frac{v^2}{w^2} = 0{,}97621;\ 0{,}90486;\ 0{,}78593;\ 0{,}61944;\ 0{,}40537;\ 0{,}14375$, and for

$\frac{v}{1 - \zeta \frac{v^2}{w^2}} = 0;\ 4{,}268;\ 9{,}210;\ 15{,}905;\ 26{,}910;\ 51{,}393$, and $173{,}913$; from Simpson's rule the mean value is:

$= (1 \cdot 0 + 4 \cdot 4{,}268 + 2 \cdot 9{,}210 + 4 \cdot 15{,}905 + 2 \cdot 26{,}910 + 4 \cdot 51{,}393 + 1 \cdot 173{,}913) \div 3 \cdot 6$

$= \frac{532{,}42}{18} = 29{,}58$; and from this the space of descent sought:

$$s = \frac{v_n - v_0}{g} \text{ times the mean value of } \frac{v}{1 - \zeta \cdot \frac{v^2}{w^2}} = \frac{25 - 0}{31{,}25} \cdot 29{,}58 = 23{,}6 \text{ feet.}$$

The corresponding time of descent is, since the mean value of $\frac{1}{1 - \zeta \frac{v^2}{w^2}}$

$= (1.0 + 4 . 1{,}024 + 2 . 1{,}105 + 4 . 1{,}272 + 2 . 1{,}614 + 4 . 2{,}467 + 1 . 6{,}957) \div 18$
$= 1{,}747,\ t = \frac{25}{31.25} . 1{,}747 = 1{,}4$ sec.

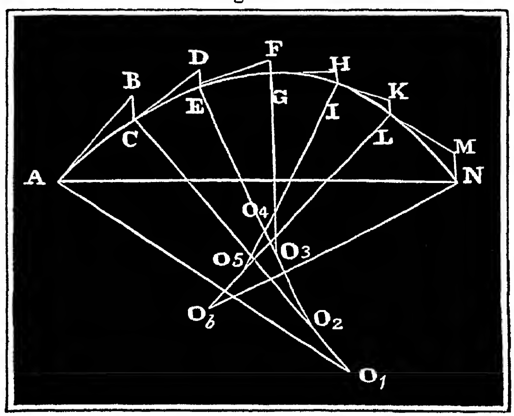
Remark. For a constant co-efficient of resistance, the higher calculus gives us:

$$v = \left(\frac{e^{\mu t} - 1}{e^{\mu t} + 1}\right) \sqrt{2g \frac{P}{\zeta F \gamma}} \text{ and } s = \frac{G}{\zeta F \gamma} L\, n \div \left(\frac{e^{\mu t} + 1)^2}{4\, e^{\mu t}}\right),$$

where $\mu = \sqrt{2\, g\, \zeta \frac{P F \gamma}{G^2}}$, e being the base of the hyperbolic system of powers, and Ln the hyperbolic logarithm.

§ 394. *Projectiles.*—We have already investigated the motion of projectiles in vacuo (§ 38), and found this motion to be parabolic; we may now obtain a more exact knowledge of motion in a resisting medium, and consider that, for instance, of a shot. In no case is the path AGN, Fig. 534, of a body passing through the air a symmetric curve; the portion GN in which the body descends is rather shorter, and, therefore, less inclined than the portion AG in which the body ascends, because the resistance of the air operating in the direction of motion tends always to shorten the portions of its path AC, CE, EG, &c., more and more; if, therefore, the first portion of the path AC, for motion in the air is only a little shorter than it would be in vacuo, the last portion LN is considerably shorter in the first motion than it is in the last. The construction of the path in a resisting medium by means of circles of curvature may be accomplised in the following manner.

Fig. 534.



From the initial velocity v_1, and the angle of elevation $BAN = \alpha_1$ it follows that the $\angle ABC = 90 - \alpha_1$, and $sin. ABC = cos. \alpha_1$, from § 40 the radius of curvature

$$O_1A = O_1C = r_1 = \frac{v_1^2}{g\ cos.\ \alpha_1},$$

hence with this we may approximately describe the portion of arc AC. If now we assume the angle subtended at the centre $AO_1\ C = \phi^0_1$, therefore $AC = s_1 = r_1\, \phi_1$, we then obtain for the succeeding particle of space CE the angle of inclination $\alpha^0_2 = \alpha^0_1 - \phi^0_1$. Let further, the height of fall $BC = h_1$, and the measure of the retardation due to the air's resistance $\zeta . \frac{v_1^2}{2g} F \gamma$ being

$$\zeta \cdot \frac{v_1^2}{2g} \cdot \frac{F\gamma}{G} = \mu\, v_1^2, \text{ therefore } \zeta \cdot \frac{F\gamma}{2\,G} = \mu,$$

from the principle of *vires vivæ*, we then obtain for the velocity v_2 at the initial point of the second portion of arc:

$$\frac{v_2^2}{2g} = \frac{v_1^2}{2g} - h_1 - \mu\left(\frac{v_1^2 + v_2^2}{2g}\right) s_1, \text{ or } (1 + \mu s_1)\frac{v_2^2}{2g}$$

$$= (1 - \mu s_1)\frac{v_1^2}{2g} - h_1, \text{ and hence } v_2 = \sqrt{\frac{(1 - \mu s_1)\, v_1^2 - 2\,g\,h_1}{1 + \mu s_1}}.$$

Since now the height of fall $h_1 = \frac{1}{2} g\tau^2 = \frac{1}{2} g\left(\frac{s_1}{s_1}\right)^2$, it follows that:

$$v_2 = \sqrt{\frac{(1 - \mu s_1)\, v_1^2 - \left(\frac{g s_1}{v_1}\right)^2}{1 + \mu s_1}} = v_1 \sqrt{\frac{1 - \mu r_1 \phi_1 - \frac{\phi_1^2}{cos.\ a_1^2}}{1 + \mu r_1 \phi_1}}.$$

If we substitute these values of α_2 and v_2 in the equation: $r_2 = \frac{v_2^2}{g\,cos.\,\alpha_2}$, we then obtain the radius of curvature $O_2C = O_2E$ of the succeeding portion of arc CE, and if we assume an angle of revolution $CO_2E = \phi_2$, it again follows from this that the angle of inclination in the vicinity of $E : \alpha_3 = \alpha_2\, \phi_2$, and the velocity at this point

$$v_3 = v_2 \sqrt{\frac{1 - \mu r_2\, \phi_2 - \frac{\phi_2^2}{cos.\ \alpha_2^2}}{1 + \mu r_2\, \phi_2}}.$$

It is therefore easy to see how the entire path of the projectile may be successively composed of circular arcs.

Example. A cast iron ball, of 4 inches diameter, is shot off at an angle of elevation of 50° with a velocity of 1000 feet, required its path, if only approximately, according to Prussian weights and measures. The radius of curvature of the first portion of arc is $r_1 = \frac{v_1^2}{g\ cos.\ \alpha} = \frac{1000000}{31{,}25\,cos.\,50°} = 49783$ ft. As the density of the air $= 0{,}0859$, and that of cast-iron $= 470$ lbs., we have then $\mu = \zeta \cdot \frac{F\gamma}{2\,G} = \zeta \cdot \frac{3\,.\,3\,.\,0{,}0859}{4\,.\,470} = 0{,}00041122$ $. \zeta$; now for $v = 1000$, $\zeta = 0{,}90$, hence $\mu = 0{,}0003701$. If we take an arc of 1° only, we then obtain the velocity at the end of it:

$$v_2 = 1000 \sqrt{\frac{1 - 0{,}0003701\,.\,49783\,.\,0{,}017453 - (0{,}017453 \div cos.\,50°)^2}{1 + 0{,}0003701\,.\,49783\,.\,0{,}017453}}$$

$= 7697$ feet.
and the radius of curvature for a second portion of arc:

$$r_2 = \frac{(769{,}7)^2}{31{,}25\,cos.\,49°} = 28897 \text{ feet.}$$

For $v_2 = 769{,}7$ feet, $\zeta = 0{,}81$, therefore $\mu = 0{,}0003331$. If, therefore, we describe with the last radius, an arc $\phi_2 = 2°$, the velocity at its ending point will be

$$v_3 = 769{,}7 \sqrt{\frac{1 - 0{,}33598 - 0{,}002831}{1{,}33598}} = 541{,}47 \text{ feet.}$$

For a third arc Q_3, the radius of curvature $r_3 = 13757$ feet, and if, therefore, we assume $\zeta = 0{,}75$, we shall then obtain at the end of a length of arc of 4°, the velocity $v_4 = 398{,}85$ feet. The radius of curvature for a fourth arc may be likewise found $r_4 = 6960{,}5$ by assuming $\zeta = 0{,}72$, and we shall then obtain the velocity $v_5 = 288{,}85$ feet, at the end of an arc of 8°, from which a fifth radius of curvature $r_5 = 3259$ feet may be calculated. Proceeding in this manner, we shall obtain, by degrees, the collective elements for the construction of the line of projection in question.

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U.

V.

W.

END OF VOL. I.

ERRATA.

Page 30, line 5 from bottom, *for* 32,22 *read* 32,2.

" 31 " 6 *for* 15,625 *read* 16,1, and *for* 250 *read* 257,6.

" " " 8 *for* 0,016 *read* 0,0155.

" " " 10 from bottom, *for* 241 *read* $241\frac{1}{2}$.

" 32 " 9 from bottom, *for* 0,480 *read* 0,465, *for* 0,320 *read* 0,310, and *for* 0,160 *read* 0,155.

" " " 11 from bottom, *for* 0,032 *read* 0,031, and *for* 0,480 *read* 0,465.

" 38 " 9 *for* $15\frac{5}{8}$ feet *read* $15\frac{5}{8}$ feet Prussian measure.

" " " 23 from top, *for* $c_1\, t_1, c_2\, t_1$ *read* $c_1\, t,\ c_2\, t$.

" 55 " 5 *for* $\frac{205}{502}$ *read* $\frac{205}{485{,}8}$, *for* 0,4083 *read* 0,4219, also *for* 0,4083 *read* 0,4219, and *for* 705,54 *read* 729,04.

" 73 " 4 from top, *for* Fig. 40 *read* Fig. 41, and in line 5, *for* Fig. 41 *read* Fig. 40.

" 75 " 3 *for* $(h-h_1)\, M$ *read* $(h-h_1)\, G$.

" " " 6 *for mass read* weight.

" 100 " 17 from bottom, *for* $x_3\, y_2\, z_1$ *read* $x_2\, y_3\, z_1$.

" 153 " 9 from bottom, *for* GOK *read* GOQ.

" 163 " 14 from bottom, *for* $\mp Q \cos. \beta$ *read* $\mp \nu\, Q \cos. \beta$.

" 184 " 10 from bottom, *for* 1 *read* l, and *for* $\lambda\, E$ *read* $\frac{\lambda}{l}\, E$.

" 189 " 17 from top, *for* $z_1 \,.\, F_1\, S\, z$, *read* $z_1 \,.\, F\, S\, z_1$.

" 257 " 13 from top, *for* $\cos. \alpha$ *read* $\operatorname{cotg}. \alpha$.

" 264 " 2 from top, *for* G *read* g.

" 331 " 6 from top, *for* shoared *read* shored.

" 394 " 15 from bottom, *for* $\boxed{1\frac{1}{4} \mid 1 \mid 2}$ *read* $\boxed{1\frac{1}{4} \mid 1\frac{1}{2} \mid 2}$.

" 414 " from bottom, *for* 01,274 *read* 0,1274.

" 422 " 15 from bottom, *for* $h = r\, t, =$ *read* $h = r, t =$.

" 426 " 12 from bottom, *for* 63,29 *read* 63,89.

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